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Mathematical Reviews

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FOUNDATIONS

***Mostowski, Andrzej. Logika Matematyczna. [Mathematical Logic].** Monografie Matematyczne, vol. 18. Warszawa-Wrocław, 1948. viii+388 pp.

This is an exceptionally good handbook of mathematical logic. It covers a great deal of material and shows many applications of mathematical logic to mathematical problems, and will generally be useful to students of mathematics rather than of philosophy. The book is written in the tradition of the Polish school, but presents a well-rounded view of the field. The exposition is easy at first and becomes progressively more difficult and more exact. After presenting sentential calculus and quantification the author explains the theory of classes and relations and follows Frege's deduction of arithmetic from class theory. Then several known resolutions of the class antinomies are discussed (e.g., types, Quine's method). Almost half the book deals with methodological problems: characteristics of formal systems, definitions, models, problems of consistency, independence, completeness, decision procedure, Gödel's method of arithmetisation, semantical antinomies, Gödel's proof of the completeness of functional calculus, and Gödel's incompleteness theorem discussed together with Tarski's theorem of the indefinability of the notion of satisfaction and with Richard's antinomy. The author is eager to persuade a mathematician that logic can be useful in his work.

H. Hiż (Cambridge, Mass.).

Carruccio, Ettore. Il problema dell'esprimibilità in simboli di un sistema ipotetico-deduttivo. Sigma no. 6–7, 357–367 (1948).

This is similar to a previous paper by the same author [Atti Soc. Nat. Mat. Modena **78**, 91–92 (1947); these Rev. **9**, 559]. It is shown in greater detail that a formal logical system having both axioms and rules of procedure (such as substitution rules and rules of inference) cannot be completely represented by means of a formalism based only on a finite number of symbols. *O. Frink.*

Carruccio, Ettore. Considerazioni sulla compatibilità di un sistema di postulati e sulla dimostrabilità delle formule matematiche. Pont. Acad. Sci. Acta **10**, 21–41 (1946).

The incompleteness theorem of Gödel states that the consistency of a certain type of formal logical system cannot be proved by means which may be formalized within the system. The author discusses some consequences of this result, such as examples of undecidable propositions. These examples include a real number which cannot be proved either to be rational or irrational, two real numbers which cannot be proved either to be equal or unequal, and an infinite series which cannot be proved either to be convergent or divergent, if the proofs are restricted to the formal system in question. *O. Frink* (State College, Pa.).

Carruccio, Ettore. I fini dei "Calculus ratiocinator" di Leibniz, e la logica matematica del nostro tempo. Boll. Un. Mat. Ital. (3) **3**, 148–161 (1948).

Robinson, Raphael M. Recursion and double recursion. Bull. Amer. Math. Soc. **54**, 987–993 (1948).

Using the methods and results of his previous paper [same Bull. **53**, 925–942 (1947); these Rev. **9**, 221] the author simplifies the proof of R. Péter that a certain function, defined by double recursion, is not primitive recursive. Furthermore, he defines by double recursion a function $G(n, x)$ and a primitive recursive function $H(n, x)$ such that, if $F(x)$ is primitive recursive, there exists a number n such that $F(x)=G(n, H(n, x))$.

A. Heyting (Amsterdam).

Halldén, Sören. A note concerning the paradoxes of strict implication and Lewis's system S1. J. Symbolic Logic **13**, 138–139 (1948).

It is known that in the simple system of strict implication S1 of C. I. Lewis, the formula (1) $\Diamond(pq)\dashv\Diamond p$ is not deducible. It follows that some of the so-called paradoxes of strict implication that arise in other systems of Lewis are not deducible in S1. The author shows, however, that certain analogues of these paradoxes are so deducible, namely (2) $\sim\Diamond p.\supset.p\dashv q$ and (3) $\sim\Diamond\sim p.\supset.q\dashv p$. These have a symbol of material implication $\supset$ replacing a symbol of strict implication $\dashv$ in the original formulas. An analogue of (1), namely (4) $\Diamond(pq)\supset\Diamond p$, is also deducible in S1.

O. Frink (State College, Pa.).

Wang, Hao. A new theory of element and number. J. Symbolic Logic **13**, 129–137 (1948).

The axioms *100–*105 and *201, *202 of Quine's "Mathematical Logic" [New York, 1940; these Rev. **2**, 65], with *200 omitted, lead to a basal logic free of contradiction, but too weak to support an arithmetic of the natural numbers. Quine's original elementhood axiom *200 is known to be too strong. The author develops an arithmetic based on Quine's axioms with *200 replaced by the axiom †*300* $(x)(y)\,x\cup y\in V$ (that is, the union of two unit classes is always an element). This is weaker than other axioms proposed by Quine to replace *200 [J. Symbolic Logic **6**, 135–149 (1941); these Rev. **3**, 289]. The author's theory of the natural numbers results from identifying the null class Λ with the number 0, and the successor of a natural number with its unit class. This differs from Quine's treatment of the natural number n as the class of all classes with n members. Quine's theory of "counter sets" also has to be modified.

This axiom system is so weak that it does not insure the existence of elements with more than two members, although there are infinitely many elements and a form of the axiom of infinity is derivable. However, this weakness has the advantage that it allows the construction of various independence models. By means of these models the author shows the independence of the axioms *201, *202, and †*300*, and also of two axioms EA2 and EA3 of Quine's paper.

O. Frink (State College, Pa.).

Ridder, J. Über mehrwertige Aussagenkalküle und mehrwertige engere Prädikatenkalküle. I. Nederl. Akad. Wetensch., Proc. 51, 670–680 = Indagationes Math. 10, 221–231 (1948).

Ridder, J. Über mehrwertige Aussagenkalküle und mehrwertige engere Prädikatenkalküle. II. Nederl. Akad. Wetensch., Proc. 51, 836–845 = Indagationes Math. 10, 264–273 (1948).

The author gives several equivalent axiom systems for an abstract many-valued logic of propositions $K^{(2)}$ with the same formal properties as the four-valued logic of S. A. Kiss [Transformations on Lattices and Structures of Logic, New York, 1947; these Rev. 9, 76]. This logic has two sum operations, two product operations, four implication operations, and four truth-values which the author calls modalities. The system has, in addition to the four-element interpretation of Kiss, further interpretations with 2^{n+m} elements, in terms of a direct product of two Boolean algebras with 2^n and 2^m elements, respectively.

In part II he further generalizes the system so that it contains an arbitrarily large number of modalities, and of sum, product and implication operations, using as a model the direct product of more than two Boolean algebras. He also constructs in the same way a many-valued calculus of functions by the introduction of quantifiers of different kinds corresponding to the different kinds of operation.

O. Frink (State College, Pa.).

Lacombe, Daniel. Sur la méthode extensive en métamathématique. Revue Sci. 85, 515–518 (1947).

En utilisant la théorie des structures de Bourbaki on peut définir plusieurs notions métamathématiques à l'aide des notions habituelles de la théorie des ensembles. Discussion des avantages méthodologiques d'une telle méthode.

A. Heyting (Amsterdam).

***Le Lionnais, F., editor. Les Grands Courants de la Pensée Mathématique.** Cahiers du Sud, 1948. 533 pp. (4 plates). 840 francs.

This symposium, dedicated to the clarification of scientific humanism, consists of some fifty articles on various aspects of mathematics and its significations. The articles range from Bourbaki on the architecture (structures!) of mathematics to Le Corbusier on the mathematical spirit in architecture (structures again!); they have been elegantly interrelated by the editor. The first section begins with a discussion by Borel of the "biggest number that can be written on the broad side of a barn door"; many striking aspects of mathematical disciplines are then presented in a mature fashion, which should be illuminating both to the layman and the professional. The section ends with a somewhat one-sided discussion of probability (but no statistics!) and its foundations. The second section, on the development of mathematics, is of especial interest to mathematicians, since it contains a number of penetrating articles such as those by É. Cartan on S. Lie, J. Dieudonné on D. Hilbert, and A. Weil on the future of mathematics.

The third section is considerably more discursive since it deals with the cultural, philosophical and scientific aspects of mathematics. There are several papers on mathematical philosophy, notably a posthumous article by L. Brunschvicg. The emphasis, however, seems curious: R. Dugas deems it necessary to show that mathematics has not been debased in importance by Bergson's intuitionism, but none of the articles display any comprehension of the real role of symbolic logic in the foundations. P. Laberenne presents a Marxist program for the resolution of the present crisis in mathematics. There are a number of articles on the relation of mathematics to physics, to technology, and to the arts, such as for instance an elegantly illustrated article by the editor on classical and romantic beauty in the facts and the methods of mathematics. In spite of the various omissions and occasional distortions noted, this volume as a whole achieves its objectives of providing a mature and stimulating presentation of the leading ideas of mathematics.

S. MacLane (Chicago, Ill.).

***Schillinger, Joseph. The Mathematical Basis of the Arts.** Philosophical Library, New York, N. Y., 1948. x+696 pp. (1 plate). $12.00.

ALGEBRA

Wiegmann, N. A. Normal products of matrices. Duke Math. J. 15, 633–638 (1948).

If A is a square matrix over the complex field, then (1) $A = HU$, where H is a positive Hermitian matrix and U is a unitary matrix. If A is a normal matrix, $AA^* = A^*A$, then the Hermitian polar matrix H and the unitary polar matrix U in the polar representation (1) of A commute. The author proves the following theorems. If A, B and AB are normal matrices, BA is a normal matrix; a necessary and sufficient condition that the product of two normal matrices be normal is that each commute with the Hermitian polar matrix of the other. He also determines necessary and sufficient conditions that a product AB of two arbitrary square matrices be a normal matrix and concludes by finding a canonical form, under a unitary similarity transformation, for a set of normal matrices which form a group under multiplication. The proofs are simple and are by induction on n, the order of the square matrices.

J. Williamson (Flushing, N. Y.).

Parker, W. V. Sets of complex numbers associated with a matrix. Duke Math. J. 15, 711–715 (1948).

Let A be a square matrix of order n with complex elements, and x and y variable complex vectors of dimension n such that $x\bar{x}' = y\bar{y}' = 1$. The set S_1 of all complex numbers $xA\bar{y}'$ has been shown by the author [same J. 12, 519–526 (1945); these Rev. 7, 107] to be precisely the set of all complex numbers in or on the circle of radius ρ_n about zero in the complex plane, where ρ_n^2 is the largest of the characteristic roots of $A\bar{A}'$. One can also characterize S_1 as the set of elements of all matrices $UA\bar{V}'$, where U and V are unitary. The subset S_2 of S_1 consisting of all complex numbers $xA\bar{x}'$ was shown by Hausdorff [Math. Z. 3, 314–316 (1919)] to be a closed convex set in the complex plane. The set S_2 is precisely the set of all diagonal elements of all matrices of the form $UA\bar{U}'$, where U is unitary. The author investigates these and related sets and proves a number of miscellaneous theorems of which the following is typical. The nondiagonal elements of all unitary transforms of an Hermitian matrix A consist of all complex numbers γ such

that $|\gamma| \leqq (\lambda_n - \lambda_1)/2$, where λ_1 is the least and λ_n is the greatest of the characteristic roots of A. *N. H. McCoy.*

Brauer, Alfred. Limits for the characteristic roots of a matrix. III. Duke Math. J. 15, 871–877 (1948).

This is a direct continuation of papers I and II [same J. 13, 387–395 (1946); 14, 21–26 (1947); these Rev. 8, 192, 559]. If ω is a characteristic root of the square matrix A of order n and x is any characteristic vector belonging to ω so that $Ax = \omega x$, then $f(A)x = f(\omega)x$, where $f(y)$ is an arbitrary polynomial. Using this fact the author deduces from theorems in papers I and II the following more general theorem. Let $A = (a_{ij})$ be a square matrix of order n and $f_1(y), \cdots, f_n(y)$ arbitrary polynomials. Let $f_i(A) = (a_{ij}^{(i)})$ and let

$$\sum_{j=1,\ j\neq i} |a_{ij}^{(i)}| = P_i^{(i)}, \quad i, j = 1, 2, \cdots, n.$$

Each characteristic root ω of A satisfies at least one of the n inequalities $|f_r(\omega) - a_{rr}^{(r)}| \leqq P_r$ and at least one of the $n(n-1)/2$ inequalities

$$|f_r(\omega) - a_{rr}^{(r)}|\ |f_s(\omega) - a_{ss}^{(s)}| \leqq P_r^{(r)} P_s^{(s)}, \quad r, s = 1, 2, \cdots, n;\ r \neq s.$$

The author illustrates the use of this theorem by applying it to find the limits between which the characteristic root of greatest absolute value of a particular five-rowed numerical matrix lie. He concludes by pointing out that theorem 6 of paper I is not true without some restrictions on the matrix A and gives the necessary corrections. *J. Williamson.*

Segre, B. Gli automorfismi del corpo complesso, ed un problema di Corrado Segre. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 3, 414–420 (1947).

Der Frage nach der Existenz von collinearen Abbildungen im Complexen, die von den Homographien und Antihomographien verschieden sind, entspricht die Frage nach der Existenz von Isomorphismen des komplexen Körpers, die von der Identität und dem Übergang zur konjugiert complexen Grösse verschieden sind. Die Existenz solcher Isomorphismen wird erschlossen unter Heranziehung der Steinitz'schen Erweiterungstheorie [Steinitz, Algebraische Theorie der Körper, de Gruyter, Berlin, 1930] mit mengentheoretischen Betrachtungen. Der Wohlordnungssatz und die Beziehungen zwischen den Mächtigkeiten der abzählbaren Mengen, des Continuums und der Menge der Funktionen ist dabei wesentlich. Es wird bewiesen, dass die Menge der isomorphen Abbildungen des komplexen Körpers und bereits schon der involutorischen unter ihnen von der Mächtigkeit der Menge der Funktionen ist.

R. Moufang (Frankfurt am Main).

Segre, B. Sulle irrazionalità da cui può farsi dipendere la determinazione di S_k appartenenti a varietà intersezioni complete di forme. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 4, 149–154 (1948).

Some years ago, the author and the reviewer found independently that a system of homogeneous equations $f_1 = 0$, $f_2 = 0, \cdots, f_k = 0$ of degrees $n_1, n_2, \cdots, n_k$ in r unknowns with coefficients in a field K has a nontrivial solution in a suitable soluble extension field K^*, provided that the number r of unknowns lies above a certain integer m depending on $n_1, n_2, \cdots, n_k$. The author here proves a generalization given by the reviewer [Bull. Amer. Math. Soc. 51, 749–755 (1945); these Rev. 7, 108]. The proof gives a recursive construction of a possible value of m and of a field K^*.

R. Brauer (Ann Arbor, Mich.).

Abstract Algebra

***Bourbaki, N. Éléments de mathématique. VII. Première partie: Les structures fondamentales de l'analyse. Livre II: Algèbre. Chapitre III: Algèbre multilinéaire.** Actualités Sci. Ind., no. 1044. Hermann et Cie., Paris, 1948. ii+157+ii pp.

This continuation of Bourbaki's compendium of all abstract mathematics [cf. these Rev. 9, 406 for Chapter II] treats tensor products, tensor spaces, Grassmann algebra and determinants; it is of especial interest in view of the paucity of conceptual and systematic accounts of these topics. The tensor product $E \otimes F$ of two modules E and F over a commutative ring A and the canonical mapping φ of the Cartesian product $E \times F$ in $E \otimes F$ are defined; any bilinear mapping f of $E \times F$ in an A-module N can then be represented as a composite $f = g\varphi$, with g a linear mapping, of $E \otimes F$ in N, and this fact is used to characterize the tensor product. Tensor products of Abelian groups and of algebras are considered. Following Whitney [Duke Math. J. 4, 495–528 (1938)] the space of tensors p times contravariant and q times covariant over a given A-module E is then characterized conceptually as the tensor product of E (p times) with the dual of E (q times). Various properties are derived, with attention to the special cases when E has a finite basis and when A is a field. The p-fold exterior power of E, whose elements are the p-vectors over E, is introduced as a suitable quotient of the p-fold tensor power of E, and characterized by the fact that alternating multilinear mappings on E^p correspond canonically to linear mappings defined on the p-fold exterior power. The direct sum of all exterior powers, with the appropriate exterior product, leads to the Grassmann algebra over E. Linear mappings on E induce linear mappings on the Grassmann algebra; these are used to give an elegant conceptual treatment of the determinant of a linear mapping. The dual of the pth exterior power of E is determined, when E has a finite basis, as the space of p-forms on E. Appendices treat infinite tensor products, tensor products of modules over noncommutative rings and the "universal mapping" question.

S. MacLane (Chicago, Ill.).

Čebotarëv, N. G. Zolotarëv's development of ideal theory. Uspehi Matem. Nauk (N.S.) 2, no. 6(22), 52–67 (1947). (Russian)

Amitsur, A. S. A generalization of a theorem on linear differential equations. Bull. Amer. Math. Soc. 54, 937–941 (1948).

Let F denote a (possibly noncommutative) field and S an automorphism of it. A right S-derivation is defined to be a mapping D of F into itself such that $(a+b)D = aD + bD$ and $(ab)D = (aS)(bD) + (aD)b$ for $a, b \in F$. The elements c with $cD = 0$ form a subfield C. The author proves that any right equation $a_n(zD^n) + a_{n-1}(zD^{n-1}) + \cdots + a_0 z = 0$ has at most n linearly independent solutions over C. Using this theorem, with the derivation $x \to xa - ax$, he then proves that an element $a \in F$ is of degree n over the center of F, if and only if F is of degree n (either left or right) over the subring of elements which commute with a. This is a special case of a theorem of Artin and Whaples [Amer. J. Math. 65, 87–107 (1943); these Rev. 4, 129]. *G. Whaples.*

Wang, Shianghaw. A counter-example to Grunwald's theorem. Ann. of Math. (2) 49, 1008–1009 (1948).

The author proves quite simply that the rational field has no cyclic extension of degree 8 for which the corre-

sponding extension of the 2-adic rationals is unramified of the same degree. This contradicts Grunwald's theorem, for which "proofs" have been given by Grunwald [J. Reine Angew. Math. 169, 103–107 (1933)] and by the reviewer [Duke Math. J. 9, 455–473 (1942); these Rev. 4, 72]. The author points out their errors and announces that he has found a modification of the theorem which is true.

G. Whaples (Bloomington, Ind.).

MacLane, Saunders. A nonassociative method for associative algebras. Bull. Amer. Math. Soc. 54, 897–902 (1948).

This paper is concerned with a proof of a theorem in the "Galois theory" of associative crossed product algebras which uses only elementary facts in the theory of nonassociative algebras, and which avoids the extensive computations involved in the associative proof [Eilenberg and Maclane, Trans. Amer. Math. Soc. 64, 1–20 (1948), in particular, paragraph 10; these Rev. 10, 5]. The theorem is concerned with conditions on a crossed product A with centre N, N being a normal, separable, finite extension of the base field P, which ensure that every automorphism of N over P can be extended to an automorphism of A over P. The proof given in this paper depends on the construction of a nonassociative algebra R which contains A as a subalgebra, and shows that certain inner mappings of R onto itself induce the required automorphisms in A.

D. Rees (Manchester).

Zelinsky, Daniel. Integral sets in quasiquaternion algebras. Duke Math. J. 15, 595–622 (1948).

A quasi-quaternion algebra over a field F is defined as follows. First construct the algebra $F(u)$ of order 2 over F by adjoining u to F, u satisfying $u^2=r$, r in F. Then the quasi-quaternion algebra $Q(r, s)$ is the set of all elements of the form $y+zv$, y and z in $F(u)$, with multiplication being defined by $(y_1+z_1v)(y_2+z_2v)=y_1y_2+z_1z_2's+(y_1z_2+z_1y_2')v$, where s belongs to F and, if $z=a+bu$, $z'=(a+b)-bu$. Further, s can always be taken as 0, 1, or a nonsquare of F without loss of generality.

This paper is concerned with the integral sets of $Q(r, s)$ when F is a Noether field, in particular when F is either the field of rational numbers or a p-adic field. The author first considers the problem of defining integral sets in $Q(r, s)$ and shows that if it is first required that an integral set S of $Q(r, s)$ should be a subring of $Q(r, s)$ containing both the ring G of integers in F and a base of Q, then the definition can be completed by any one of the following five conditions: (1) every element of S has an integral characteristic right function; (2) as (1) with "characteristic" omitted; (3) S is a finite G-module; (4) as (1) with "left" replacing "right"; (5) as (2) with "left" replacing "right." These five definitions are equivalent.

In the following section, the maximal integral sets of $Q(r, s)$ are determined in the case where F is a p-adic field, r is an integer of F, and $s\neq 0$ (s may then be taken as a square-free integer). The results are as follows: $S_0=G+Gu+Gv+Guv$ is always a maximal integral set and is the only such unless both $p=2$ and $s\equiv 1 \pmod 4$. In the latter case, there are exactly 2 maximal integral sets if $s\equiv 5 \pmod 8$ and 4 if $s\equiv 1 \pmod 8$. From these results, it follows, as in the associative case, that, if F is the rational field, $Q(r, s)$ has either 1, 2, or 4 maximal integral sets.

Next, F still being a p-adic field, the ideal theory of $S=S_0$ is studied. In this theory the number $s(4r-1)$ acts as a kind of discriminant. If it is prime to p, then the only ideals of S are the powers of pS. If only s is assumed to be prime to p, then there are m reduced ideals in S if s is a nonsquare in G and $\frac{1}{2}m(m-1)$ if s is a square in G. Here p^m is the greatest power of p dividing $4r-1$ and reduced ideals are those not divisible by p. The author finally considers the case where s is divisible by p, and enumerates the reduced ideals, which fall into 6 categories, which, however, are not disjoint.

In considering multiplication of ideals, the author points out that the product of two ideals in the nonassociative case need not be an ideal. He therefore introduces a $\times$-product of two ideals, defined as the smallest ideal containing the product, and uses this product where necessary. In studying the product theory of ideals, the author considers 5 cases. The first, where $s(4r-1)$ is prime to p, has already been mentioned. The next two, completing the case s prime to p but distinguishing the cases where s is or is not a quadratic residue of p, both lead to a product theory in which the ideals form a commutative semi-group but whose structure is more complicated than in the first case. If s is divisible by p, the ideals no longer necessarily form a semi-group under multiplication, the product of ideals being neither associative nor commutative in general.

In the last section of the paper the above theory is applied to the theory of valuations of $Q(r, s)$. The particular case of interest is that where F is a p-adic field, and the valuation of $Q(r, s)$ induces the usual valuation in F. As in the associative case, the set of elements of $Q(r, s)$ which are strictly positive with respect to the valuation form a maximal ideal of the integral set S_0. By considering the various possibilities for this maximal ideal, the author shows that necessary and sufficient conditions for $Q(r, s)$ to possess a valuation extending that of F is that r should not be a quadratic residue of p or be divisible by p and that s is also not a quadratic residue of p but may be divisible by p. A corresponding result is deduced for the case where F is the field of rationals.

D. Rees (Manchester).

Mal'cev, A. I. On normed Lie algebras over the field of rational numbers. Doklady Akad. Nauk SSSR (N.S.) 62, 745–748 (1948). (Russian)

The algebras in question may be infinite-dimensional. The norm satisfies the following axioms: $0<|a|\leq\frac{1}{2}$ for $a\neq 0$, $|a+b|\leq\max(|a|, |b|)$, $|ab|\leq|a|\cdot|b|$, and $|\lambda a|=|a|$ for any rational λ. These algebras are precisely the inverse limits of discrete nilpotent Lie algebras over the rational numbers. A connection is established between such algebras and inverse limits of nilpotent divisible groups with no elements of finite order.

I. Kaplansky (Princeton, N. J.).

THEORY OF GROUPS

Kolchin, E. R. On certain concepts in the theory of algebraic matric groups. Ann. of Math. (2) 49, 774–789 (1948).

This is a continuation of a paper [same vol., 1–42 (1948); these Rev. 9, 561] by the author. The main results of the present paper are the following. Every anticompact (respectively, quasicompact) group is reducible to special triangular (respectively, diagonal) form. Any Abelian matric group is representable in one and only one way as the direct product of a quasicompact and an anticompact group which are

simultaneously reducible, the first to diagonal form, the second to special triangular form. If σ is a generic point of a connected matric group G and k an integer not divisible by the characteristic of the basic field, then σ^k is a generic point of G. The notion of connectedness of an algebraic matric group G cannot be defined in general in terms of the abstract group structure of G; but this becomes possible if the characteristic of the basic field is 0, in which case G is connected if and only if it has no subgroup of finite index greater than 1. Quasicompactness is not definable in terms of the abstract group structure (even over fields of characteristic 0), but becomes such if it is assumed that the group is connected and its dimension is given. In that case, the criterion of quasicompactness is given in terms of the order of magnitude of the number of elements of the group whose orders are finite and divide a given k.

C. Chevalley (Paris).

Neumann, Hanna. Generalized free products with amalgamated subgroups. Amer. J. Math. **70**, 590–625 (1948).

Let U be a group, let G_α be a set of groups (finite or infinite in number), and suppose that for each α there is an isomorphism σ_α of U onto a subgroup U_α of G_α. Let F be the free product of the groups G_α, let H be the smallest normal subgroup of F which contains the products $\sigma_\alpha u(\sigma_\beta u)^{-1}$ for all α and β, and for all $u \in U$. The factor group $G = F/H$ is called the free product of the groups G_α with the amalgamated subgroup U. The author's major goal, determining the structure of all subgroups of G [to be done in a later paper] gives rise to a preliminary problem of determining existence conditions for a more general type of free product. Now, let the given groups G_α be such that each pair G_α and G_β contain subgroups $U_{\alpha\beta}$ and $U_{\beta\alpha}$, respectively isomorphic under a mapping $\sigma_{\alpha\beta}: u_{\alpha\beta} \to u_{\beta\alpha} = \sigma_{\alpha\beta}(u_{\alpha\beta})$ $[\sigma_{\alpha\beta}^{-1} = \sigma_{\beta\alpha}]$. Let K be the smallest normal subgroup of F which contains the products $u_{\alpha\beta}^{-1}\sigma_{\alpha\beta}(u_{\alpha\beta})$ for all α, all β and all $u_{\alpha\beta} \in U_{\alpha\beta}$. If $G = F/K$ contains subgroups G_α^* isomorphic to the G_α such that $G_\alpha^* \cap G_\beta^*$ is isomorphic to $U_{\alpha\beta}$, then G is called the free product of the G_α with amalgamated subgroups $U_{\alpha\beta}$. Certain necessary and certain sufficient conditions are given for the existence of free products with amalgamated subgroups, with a more detailed investigation for the case of three factors. The main result of the paper is a reduction theorem which states that if U_α is the subgroup of G_α generated by all the $U_{\alpha\beta}$ then the free product of the groups G_α with amalgamated subgroups $U_{\alpha\beta}$ exists if and only if the free product of the groups U_α with amalgamated subgroups $U_{\alpha\beta}$ exists. The proof of this main theorem consists of a construction of G, and is similar in principle to O. Schreier's construction of the free product with one amalgamated subgroup [Abh. Math. Sem. Hamburg. Univ. **5**, 161–183 (1927)].

R. M. Thrall (Ann Arbor, Mich.).

Amato, V. Sui gruppi diedrali. Matematiche, Catania **2**, 58–64 (1947).

The subgroup of the octahedral group of rotations of the complex sphere which leaves invariant the polyhedron with vertices ± 1, $\pm i$, 0, ∞ is the dihedral group of order 8 and is the normalizer of one of the elements. The equation of the polyhedron is given [as in L. Bianchi, Lezioni sulla teoria dei gruppi di sostituzione e delle equazione algebriche secondo Galois, Pisa, 1900] and the critical points are found.

H. A. Thurston (Bristol).

Kaplan, Samuel. Extensions of the Pontrjagin duality. I. Infinite products. Duke Math. J. **15**, 649–658 (1948).

On dit que deux groupes topologiques abéliens sont en dualité si chacun est le groupe des caractères de l'autre (la topologie sur le groupe des caractères de G étant celle de la convergence uniforme sur toute partie compacte de G). Soient (G_λ) et (H_λ) deux familles de groupes, G_λ et H_λ étant en dualité pour chaque λ; soit G le produit direct des G_λ, muni de la topologie de Tychonoff; soit H le produit faible des H_λ (au sens de Lefschetz); l'auteur démontre que, si l'on introduit sur H une topologie convenable, G et H sont en dualité. Il s'ensuit évidemment que le résultat bien connu de Pontrjagin est valable, par exemple, pour tout produit direct de groupes localement compacts.

Le groupe des nombres réels modulo 1 joue dans les démonstrations un rôle assez épisodique; on pourrait probablement lui substituer le groupe des nombres réels, par exemple: ce qui peut être utile en topologie algébrique.

R. Godement (Nancy).

Lee, H. C. Tensor invariants of Lie groups. Duke Math. J. **15**, 639–647 (1948).

Let G be a connected Lie group. Then the left translations in G allow one to associate to every tensor T_s at the unit element of G a tensor field T_s on G which is invariant by the left translations, and every left invariant tensor field may be obtained in this way. The components of T_s with respect to a basic system of left invariant, covariant and contravariant vectors are constant. This simple fact is formulated and proved in a complicated tensor notation. The condition for T_s to be right invariant is formulated in the same complicated notation. It is proved that the maximum number of closed invariant differential forms of a given degree p which are linearly independent modulo the space of exact forms is the same whether one considers left invariance or right invariance; it is erroneously asserted that this number is a topological invariant of the group manifold [this is not true in general for noncompact groups]. It is proved that a differential form which is invariant on both sides is closed.

C. Chevalley (Paris).

Zelinsky, Daniel. On ordered loops. Amer. J. Math. **70**, 681–697 (1948).

After some preliminaries on topological loops in general, the author confines his attention to simply ordered loops, i.e., loops in which an order relation $x<y$ is defined satisfying (i) $x<y$ is transitive; (ii) one and only one of the relations $x<y, x=y, x>y$ holds; (iii) $x<y$ implies $x+z<y+z$ and $z+x<z+y$. (The loop operation is written as addition but is not necessarily commutative.)

Many of the elementary properties of simply ordered groups [cf., for example, Birkhoff, Ann. of Math. (2) **43**, 298–331 (1942); these Rev. **4**, 3] carry over to loops without alteration. In particular, the author considers l-ideals of a loop, defined as normal sub-loops containing with x, y all elements between x and y. As in the case of simply ordered groups these are characterised as the kernels of order-preserving homomorphisms and are simply ordered with respect to inclusion.

A simply ordered loop L has a natural topology induced by taking the open intervals, i.e., the sets of elements lying between two elements of L, as a base of neighbourhoods. With this topology, the loop is a topological loop, and the topology is discrete if and only if it contains a least positive element, which then belongs to the centre. Further, if the

intersection of all l-ideals contains only the identity 0, the l-ideals form a base of neighbourhoods of the identity. In the contrary case, the intervals $y<x<z$, where y, z belong to the intersection M_0 of all l-ideals, and satisfy $y<0<z$, form a base of neighbourhoods of the identity.

For the rest of the paper, the author restricts his attention to loops such that the centre contains a subgroup G of finite index relative to the whole loop. He proves that, unless L consists of the identity 0 alone, this condition on L implies the existence of a nonzero l-ideal in the centre. From this it follows that there exists a finite chain $0=M_0\subset\cdots\subset M_n=L$ of l-ideals of L with the properties (i) M_i is an l-ideal of L properly containing M_{i-1}, (ii) M_i/M_{i-1} is in the centre of L/M_{i-1}, (iii) M_i is maximal with respect to (i) and (ii). Further, if G is the subgroup of the centre of L of finite index in L mentioned above, $K=L/G$ possesses an analogous series $0=H_0\subset\cdots\subset H_n=K$, the H_i, however, being normal sub-loops of K. These results are used to prove that G and L have the same order-type, with the consequence that if G is Archimedean ordered then so is L.

In the last section of the paper, the theory of loop extensions is used, in conjunction with the results above, to give a construction of all simply ordered loops with a subgroup G in the centre of finite index in the loop. The construction, which involves the structure of an ordered group G, together with a sequence of finite Abelian groups which turn out to be the quotient-groups H_i/H_{i-1}, is too complicated to give here. *D. Rees* (Manchester).

NUMBER THEORY

De Backer, S. Un théorème fondamental. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **33**, 632–634 (1947).

The author proves the following theorem. Let a and m be relatively prime integers. There exists at least one solution of the congruence $ax\equiv y \pmod m$, where $0<x\leqq\sqrt{m}$ and $0<|y|\leqq\sqrt{m}$. [This theorem is not new. It is due to A. Thue, Arch. Math. Naturvid. **34**, no. 15 (1914). See also O. Ore, Number Theory and Its History, McGraw-Hill, New York, 1948, p. 268; these Rev. **10**, 100.] *A. Brauer* (Chapel Hill, N. C.).

Ballieu, Robert. Sur des congruences arithmétiques. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **34**, 39–45 (1948).

The theorem of the preceding review is generalized as follows. The congruence $ax\equiv y \pmod m$, $(a, m)=1$, has at least one solution for which $0<x<b$ and $0\leqq|y|\leqq m/b$. [This result is also not new. See, for instance, Vinogradov, Trans. Amer. Math. Soc. **29**, 209–226 (1927), or A. Scholz, Einführung in die Zahlentheorie, de Gruyter, Berlin, 1939, p. 45.] *A. Brauer* (Chapel Hill, N. C.).

De Backer, S. M. Solutions modérées d'un système de congruences du premier degré pour un module premier p. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **34**, 46–51 (1948).

The author states the following generalization of the theorem of the second preceding review without proof. If $(a, m)=1$ and A is an arbitrary integer, then there exists at least one solution of the congruence $ax\equiv A+y \pmod m$ for which $0\leqq|x|\leqq\sqrt{m}$ and $0\leqq|y|\leqq\sqrt{m}$. This theorem is not correct. Set, for instance, $a=50$, $m=101$, and $A=75$. For $0\leqq|x|<11$ we have $ax\equiv d \pmod{101}$, where $0\leqq d\leqq 5$, or $46\leqq d\leqq 55$, or $96\leqq d\leqq 101$, while $65\leqq A+y\leqq 85$ for $0\leqq|y|<11$. The proofs of the other results are based on this theorem. *A. Brauer* (Chapel Hill, N. C.).

Shapiro, Harold N. Note on a problem in number theory. Bull. Amer. Math. Soc. **54**, 890–893 (1948).

If x, a, n are integers and $x\equiv a \pmod n$, $-\frac{1}{2}n<a<\frac{1}{2}n$, then the author defines $\|x\|_n=|a|$. His main concern is with the group G_n of residues modulo n which are prime to n, and he finds sufficient conditions in order that a mapping $T(x)$ of G_n into a subset of itself should, in fact, be a homomorphic mapping of G_n into a subgroup. Two of the principal results proved are as follows. (1) If $\|T(xy)-T(x)T(y)\|_n<k$, $k<\min_{p|n} p$, and $T(x)$ takes more than $8k^2$ distinct values, then $T(xy)\equiv T(x)T(y) \pmod n$. (2) If $n=p^\alpha$ is a power of an odd prime, $\|T(xy)-T(x)T(y)\|_{p^\alpha}<k$, $k<p$, and $T(x)$ takes more than $8k^2$ distinct values, then there exists an integer a such that $T(x)\equiv x^a \pmod{p^\alpha}$. *L. Mirsky.*

Weil, André. On some exponential sums. Proc. Nat. Acad. Sci. U. S. A. **34**, 204–207 (1948).

Let k be a finite field of q elements and χ a character of the multiplicative group of k. Let $k(t)$ be the field of rational functions in one transcendental element t with coefficients in k. Further let ω be a character of the multiplicative group of power series in an indeterminate T over k, of conductor T^N and which is such that it has the value 1 for every series which reduces to a monomial cT^m. Then if $\mathfrak{D}$ is a finite divisor, rational over k, consisting of d points ξ_ν with multiplicities a_ν, and if $R_\mathfrak{D}(x)=\prod_\nu(x-\xi_\nu)^{a_\nu}$, the author shows that the inequality $|\sum\omega(1-xT)\chi[R_\mathfrak{D}(x)]|\leqq(N+d-2)\sqrt{q}$, where the summation is extended over all elements of k, different from the ξ_ν (if any of these is in k) is a consequence of the Riemann hypothesis for the function field $k(t)$ [cf. A. Weil, Actual. Sci. Ind., no. 1041 = Publ. Inst. Math. Univ. Strasbourg **7** (1945), Hermann, Paris, 1948; these Rev. **10**, 262]. This result has important consequences in number theory. If, for instance, k is a prime field of p elements, $p\neq 2$, and if ψ is a character of the additive group of k, we can take $R_\mathfrak{D}(t)=t^2-a$, $a\neq 0$, and for ω the character of conductor T^2 determined by $\omega(1+x_1T+x_2T^2+\cdots)=-\psi(x_1)$. Then (since $N=d=2$) the inequality reduces to

$$|\sum\psi(x)\chi(x^2-a)|\leqq 2\sqrt{p},$$

which, if χ is the quadratic character, is equivalent to

$$|\sum_{x=1}^{p-1}\exp(2\pi i/p)(cx+dx^{-1})|\leqq 2\sqrt{p}.$$

H. D. Kloosterman (Leiden).

Daniloff, Georges. Contribution à la théorie des fonctions arithmétiques. Sbornik Bulgar. Akad. Nauk. **35**, 479–590 (1941). (Bulgarian. French summary)

The methods employed in the paper are elementary. The first chapter is concerned with identities involving coefficients of formal Dirichlet series of the type

$$\sum_{n=1}^{\infty}Q_{k,\nu}(n)/n^s=\left(\sum_{n=1}^{\infty}F(n)/n^s\right)\left(\prod_{j=1}^{k}\sum_{n=1}^{\infty}f_j(n)/n^{s\nu}\right)^{-1},$$

where $F, f_1, \cdots, f_k$ are completely multiplicative functions. The second chapter deals with relations between certain finite arithmetic sums of the type

$$\sum_{x_k=1}^{N}\cdots\sum_{x_1=1}^{N}F(n, mx_1, \cdots, mx_k)f((n, x_1, \cdots, x_k)^{(\nu)}),$$

where $(a, b)^{(\nu)}$ denotes the largest νth power which divides the greatest common divisor (a, b). Special cases of such

sums are, for example, Ramanujan sums and certain generalizations of them. Chapter III is concerned with the asymptotic behavior and distributions of some number-theoretic functions; e.g., with the asymptotic behavior of $\sum_{n=1}^{N} n^k \varphi_{ak}^{s}(n)$, where $\varphi_{ak}^{s}(n) = \sum \cdots \sum (x_1 x_2 \cdots x_k)^a$, summed over $1 \leqq x_i \leqq n$, $(n, x_1, \cdots, x_k)^{(s)} = 1$, or of $\sum_{n=1}^{N} \tau_\nu(n)$, where $\tau_\nu(n)$ is the number of divisors of n which are νth powers. *P. Hartman* (Baltimore, Md.).

Mirsky, L. Note on a theorem of Carlitz. Duke Math. J. 15, 803–815 (1948).

Asymptotic formulae, for $n \to \infty$, are obtained for certain sums of the form

$$\sum_{n_1+\cdots+n_s=n} n_1^{\alpha_1} \cdots n_s^{\alpha_s} F_1(n_1) \cdots F_s(n_s),$$

where $\alpha_1, \cdots, \alpha_s$ are nonnegative numbers, and $F_1, \cdots, F_s$ are arithmetic functions satisfying given conditions. Included is a theorem of Carlitz [Quart. J. Math., Oxford Ser. 2, 97–106 (1931); 3, 273–290 (1932)], but with an improved estimate of the remainder term for $s > 2$. The general results are used to obtain asymptotic formulae for the following: (a) the problem of Evelyn and Linfoot [see J. Reine Angew. Math. 164, 131–140 (1931)] and Page [see J. London Math. Soc. 7, 24–27 (1932)];

(b) $$\sum_{n_1+\cdots+n_s=n} \varphi_{\alpha_1}(n_1) \cdots \varphi_{\alpha_s}(n_s),$$

where $\varphi_k(n) = n^k \prod_{p|n}(1-p^{-k})$, $k > 0$;

(c) $$\sum_{n_1+\cdots+n_s=n} \sigma_{\alpha_1}(n_1) \cdots \sigma_{\alpha_s}(n_s),$$

where $\sigma_k(n) = \sum_{d|n} d^k$. *W. H. Simons* (Vancouver, B. C.).

Erdös, P. On the difference of consecutive primes. Bull. Amer. Math. Soc. 54, 885–889 (1948).

Let p_k be the kth prime and $d_k = p_{k+1} - p_k$; also let $N(c, n)$ and $N'(c, n)$ be the numbers of integers $k \leqq n$ for which $d_{k+1} > (1+c)d_k$ and $d_{k+1} < (1-c)d_k$, respectively, and let $N_t(n)$ and $N_t'(n)$ be the numbers of integers $k \leqq n$ for which $\{\frac{1}{2}(p_{k+1}^t + p_{k-1}^t)\}^{1/t} > p_k$ and $\{\frac{1}{2}(p_{k+1}^t + p_{k-1}^t)\}^{1/t} < p_k$, respectively. The author proves that positive numbers c_1, c_2 ($c_1 < 1$, $c_2 < 1$) exist for which both $N(c_1, n)$ and $N'(c_1, n)$ are greater than $c_2 n$. From this he deduces that, for fixed t and sufficiently large n, $N_t(n)$ and $N_t'(n)$ are each greater than $\frac{1}{2}c_2 n$. This is an improvement on a result obtained by the author and P. Turán in an earlier paper [same vol., 371–378 (1948); these Rev. 9, 498] in which it was shown that $N_t(n)$ and $N_t'(n)$ tend to infinity with n. The proofs, which employ Brun's method, are elementary; there are several minor misprints. *R. A. Rankin* (Cambridge, England).

Robinson, Raphael M. The critical numbers for unsymmetrical approximation. Bull. Amer. Math. Soc. 54, 693–705 (1948).

If ξ is an irrational number, let $M^+(\xi)$ denote the least upper bound of the values of μ for which infinitely many rational numbers A/B satisfy the inequality $0 < A/B - \xi < 1/(\mu B^2)$. Similarly $M^-(\xi)$ is defined measuring the approximability from the left. The number ξ is called critical if there is no other irrational number ξ' for which simultaneously $M^+(\xi') < M^+(\xi)$, $M^-(\xi') < M^-(\xi)$. The author gives a necessary and sufficient condition that ξ be critical. This condition requires that the expansion of ξ as a continued fraction has the form

$$\xi = [q_0, q_1, \cdots, q_{n-1}, 1, r_1, 1, r_2, 1, r_3, \cdots],$$

where the sequence $r_1, r_2, \cdots$ is of a certain type. [Cf. same Bull. 53, 351–361 (1947); these Rev. 8, 566.]

J. F. Koksma (Amsterdam).

Negoescu, Nicolae. Quelques précisions concernant le théorème de M. B. Segre sur des approximations asymétriques des nombres irrationnels par les rationnels. Bull. École Polytech. Jassy [Bul. Politehn. Gh. Asachi. Iaşi] 3, 3–16 (1948).

B. Segre [Duke Math. J. 12, 337–365 (1945); ces Rev. 6, 258] a démontré géométriquement le théorème suivant: Tout nombre irrationel θ a une infinité d'approximations rationnelles p/q telles que l'on ait

$$(1) \qquad -1/(q^2\xi) < p/q - \theta < \tau/(q^2\xi)$$

où $\tau \geqq 0$ est réel et $\xi = (1+4\tau)^{\frac{1}{2}}$. C. D. Olds en donna une démonstration arithmétique pour le cas $\tau \geqq 1$ [Bull. Amer. Math. Soc. 52, 261–263 (1946); ces Rev. 8, 196]. L'auteur en donne une troisième démonstration, généralisant la méthode de Borel qui fait usage de la théorie des fractions continues et qui permet l'auteur (a) de remplacer la condition $\xi = (1+4\tau)^{\frac{1}{2}}$ par $\xi = \max((1+4\tau)^{\frac{1}{2}}, (\tau^2+4\tau)^{\frac{1}{2}})$, (b) de la remplacer par une autre condition encore meilleure, si l'on exclut les nombres θ équivalent à $(1+\sqrt{5})/2$ et (c) de démontrer des précisions sur les réduites p_n/q_n qui réalisent l'approximation (1). *J. F. Koksma* (Amsterdam).

Korobov, N. M. On functions with uniformly distributed fractional parts. Doklady Akad. Nauk SSSR (N.S.) 62, 21–22 (1948). (Russian)

A function $f(x)$ is called uniformly distributed if the sequence $\{f(k)\}$, $k = 1, 2, \cdots$, where $\{z\}$ denotes the fractional part of z, fills the interval $(0, 1)$ with asymptotically uniform density. A function $f(x)$ is called completely uniformly distributed if, for every set of integers $n_1, n_2, \cdots, n_s$, the function $n_1 f(x+1) + \cdots + n_s f(x+s)$ is uniformly distributed. Theorem 1: there exist completely uniformly distributed functions; an example is $f(x) = \sum_{k=0}^{\infty} x^k \exp(-e^k)$. Theorem 3: let $a \geqq 2$ be an integer, $f(x)$ a completely uniformly distributed function, and $b = \sum_{k=1}^{\infty} a^{-k}[a\{f(k)\}]$; then the function ba^x is uniformly distributed. Proofs of these theorems are not given, but are said to be obtainable by following the method of Weyl [Math. Ann. 77, 313–352 (1916)] and using recent results of Vinogradoff [C. R. (Doklady) Acad. Sci. URSS (N.S.) 43, 47–48 (1944); these Rev. 6, 170] and others on the order of magnitude of trigonometrical sums. *F. J. Dyson* (Princeton, N. J.).

***Kuipers, Lauwerens. De Asymptotische Verdeling Modulo 1 van de Waarden van Meetbare Functies. [The Asymptotic Distribution Modulo 1 of the Values of Measurable Functions].** Thesis, Free University of Amsterdam, 1947. vi+112 pp.

Most of the known results about asymptotic distribution deal with the case of a discrete sequence $f(1), f(2), f(3), \cdots$. The purpose of the present paper is to establish analogous results for functions $f(t)$ with a continuous range of arguments. The investigation is based on the following definition. A measurable function $f(t)$ defined for $t \geqq 0$ is said to have a uniform distribution mod 1 if, denoting by $S_T(\alpha, \beta)$ (where $T \geqq 0$, $0 \leqq \alpha \leqq \beta \leqq 1$) the set of values t satisfying $0 \leqq t \leqq T$; $\alpha \leqq f(t) < \beta \pmod 1$, one has, for every interval (α, β), $\lim_{T\to\infty} T^{-1} \operatorname{meas} S_T = \beta - \alpha$ (modifications of this definition are obtained by replacing the intervals (α, β) by countable sums of intervals or more generally by arbitrary measurable sets).

In analogy with Weyl's well-known criterion for uniform distribution of sequences, the author shows that a measurable function $f(t)$ has a uniform distribution mod 1 if and only if, for every positive integer h, $\lim_{T\to\infty} T^{-1}\int_0^T e^{2\pi i h f(t)}dt=0$. A similar result is proved for the multidimensional distribution of values of a system of functions $f_1(t), \cdots, f_n(t)$. Among the remaining results of the paper we note the following theorem. A differentiable function $f(t)$ $(t\geqq 0)$, whose derivative $f'(t)$ is a nonincreasing function satisfying $tf'(t)\to\infty$ for $t\to\infty$, has a uniform distribution mod 1. [This is a generalized analogue of a theorem due to Pólya and Szegö.] Further results are analogues of theorems of Koksma, van der Corput and Schoenberg. *W. Hurewicz.*

Cassels, J. W. S. On two problems of Mahler. Nederl. Akad. Wetensch., Proc. 51, 854–857 = Indagationes Math. 10, 282–285 (1948).

This paper answers two questions proposed by Mahler by constructing a plane star domain K which has a critical lattice with no points on the boundary of K and which possesses a group Γ of affinities which map K into itself. The group Γ has the properties that an appropriate substitution will map a point of the plane other than O into a point of the plane beyond an arbitrary distance from the origin while another appropriate substitution will map a point of K into a point within a certain fixed circle $|X|<C$. *D. Derry* (Vancouver, B. C.).

Hlawka, Edmund. Über Potenzsummen von Linearformen. II. Österreich. Akad. Wiss. Math.-Nat. Kl. S.-B. IIa. 156, 247–254 (1948).

The author considers here the minimum of $|L_1|^p+\cdots+|L_n|^p$ for $1\leqq p\leqq 2$ [see a previous paper on the case $p\geqq 2$, same S.-B. 154, 50–58 (1945); these Rev. 9, 500]. By an appropriate modification of Blichfeldt's method, he obtains an improvement on Minkowski's estimate for the minimum. *H. Davenport* (London).

Hlawka, Edmund. Über Gitterpunkte in Zylindern. Österreich. Akad. Wiss. Math.-Nat. Kl. S.-B. IIa. 156, 203–217 (1948).

Consider a cylinder in n-dimensional space, whose base is an ellipsoid in m-dimensional space $(2\leqq m<n)$. The author proves that, if the volume exceeds $(m+2)2^{n-1-\frac{1}{2}m}$, the cylinder contains a point with integral coordinates other than the origin. The proof is by the author's form of Blichfeldt's method [Math. Z. 49, 285–312 (1943); these Rev. 5, 201], and a knowledge of this is assumed. The theorem relates, more generally, to a body defined by $g(x_1, \cdots, x_m)\leqq 1$, $h(x_{m+1}, \cdots, x_n)\leqq 1$, where h is a convex function and g has a specified measure of convexity, in a sense defined by the author. Later sections of the paper are concerned with extensions of the author's "Alternativsatz." [Note by the reviewer: for the cylinder with $m=2$, see K. Mahler, Quart. J. Math., Oxford Ser. 17, 16–18 (1946); these Rev. 7, 368.] *H. Davenport* (London).

Ollerenshaw, Kathleen. On the minima of indefinite quadratic forms. J. London Math. Soc. 23, 148–153 (1948).

Let Λ be an admissible lattice for the star domain $|xy|\leqq 1$ for which the lower bound of the product $|\xi\eta|$ is 1 when (ξ, η) runs through all points of Λ other than the origin. The author locates all Λ which have minimum determinant $\sqrt{5}$ and shows the minimum determinant of all the remaining lattices Λ is $2\sqrt{2}$. By an affinity which maps the region $|xy|\leqq 1$ into itself a lattice Λ is found which has a point on the line $y=x$ within a certain neighborhood of the point (1, 1). The results are obtained by studying the intersection points of the hyperbolas $xy=\pm 1$ and the lines of minimum distance from the origin which are parallel to the line $y=x$ and contain points of the above Λ. The first two of the Markoff minima for real indefinite binary quadratic forms [Dickson, Studies in the Theory of Numbers, University of Chicago Press, 1930, chap. 7] may be determined from the above results. *D. Derry* (Vancouver, B. C.).

Delone, B. On the work of A. A. Markov "On binary quadratic forms with positive determinant." Uspehi Matem. Nauk (N.S.) 3, no. 5(27), 2–5 (1948). (Russian) Cf. the following review.

Markov, A. On binary quadratic forms with positive determinant. Uspehi Matem. Nauk (N.S.) 3, no. 5(27), 6–51 (1948). (Russian)

The work in question was originally published in St. Petersburg in 1880.

Inkeri, K. Neue Beweise für einige Sätze zum Euklidischen Algorithmus in quadratischen Zahlkörpern. Ann. Univ. Turkuensis Ser. A. 9, no. 1, 15 pp. (1948).

The paper contains new proofs of the following three theorems. (1) If a real quadratic field $R(\sqrt{m})$, where m is not divisible by any square greater than 1, has class number 1, then either m is a prime or it is a product of two different primes which are not congruent to 1 (mod 4) [Behrbohm and Rédei, J. Reine Angew. Math. 174, 192–205 (1936)]. (2) Except for $m=2, 3, 6, 7, 11, 19$, the Euclidean algorithm does not exist in $R(\sqrt{m})$, if $m=2, 3$ (mod 4) [Berg, Kungl. Fysiografiska Sällskapets i Lund Förhandlingar [Proc. Roy. Physiog. Soc. Lund] 5, 53–58 (1935); Keston, Yale University Thesis, 1935]. (3) The Euclidean algorithm does not exist in $R(\sqrt{61})$. The author simplifies a method of Hua and Shih [Amer. J. Math. 67, 209–211 (1945); these Rev. 6, 256]. *L.-K. Hua.*

Kiselev, A. A. An expression for the number of classes of ideals of real quadratic fields by means of Bernoulli numbers. Doklady Akad. Nauk SSSR (N.S.) 61, 777–779 (1948). (Russian)

Let p be a prime congruent to 1 (mod 4). Let $h(p)$ be the number of classes of ideals in the quadratic field $R(\sqrt{p})$, and let $T+U\sqrt{p}(>1)$ be the fundamental unit of the field. The author proves that $h(p)U\equiv TB_{\frac{1}{2}(p-1)}$ and

$$T\equiv(-1)^{\frac{1}{2}(h(p)+1)}(\tfrac{1}{2}(p-1))! \pmod p,$$

where $B_{\frac{1}{2}(p-1)}$ is a Bernoulli number. It is known that

$$\sqrt{p}(T+U\sqrt{p})^{h(p)}=\prod_{0<b<p}(1-\zeta^{b^2}), \tag{1}$$

where b runs over all quadratic nonresidues, modulo p, and $\zeta=e^{2\pi i/p}$. By means of Gaussian sums, we find easily that

$$S_k=\sum_b(1-\zeta^{b^2})^k=\tfrac{1}{2}p-\tfrac{1}{2}\sqrt{p}\sum_{\nu=1}^{k}(-1)^\nu\left(\frac{\nu}{p}\right)\binom{k}{\nu}. \tag{2}$$

By Waring's formula on symmetric functions, we can express the left side of (1) in terms of $S_k(1\leqq k\leqq m=\frac{1}{2}(p-1))$. Noticing that $S_1^{l_1}\cdots S_m^{l_m}\equiv 0 \pmod{p^l}$, if $l_1+\cdots+l_m>2$, we have

$$\begin{aligned}\prod_{0<b<p}(1-\zeta^{b^2}) &\equiv -\frac{S_m}{m}+\frac{1}{2}\sum_{i,j\geqq 1,\ i+j=m}\frac{S_iS_j}{ij} \pmod{p^l} \\ &\equiv 2S_m+\frac{1}{2}\sum_{i,j\geqq 1,\ i+j=m}\frac{S_iS_j}{ij} \pmod{p^l}.\end{aligned} \tag{3}$$

Since $\binom{p}{\nu}\equiv\nu^m \pmod p$, we have

$$S_k\equiv\tfrac{1}{2}p-\tfrac{1}{2}\sqrt{p}\sum_{\nu=1}^{k}(-1)^\nu\nu^m\binom{k}{\nu} \pmod{p^{\frac{1}{2}}}.$$

The higher difference $\sum_{\nu=1}^{k}(-1)^\nu\nu^m\binom{k}{\nu}$ is denoted by $(-1)^k k\Delta_{m-1}^{k-1}$. Then we have, by (1) and (3),

$$(4)\quad \sqrt{p}(T^{h(p)}+h(p)T^{h(p)-1}U\sqrt{p})\equiv p-\sqrt{p}m\Delta_{m-1}^{m-1}+\tfrac{1}{2}p\sum_{i,j\geq 1,\ i+j=m}\Delta_{m-1}^{i-1}\Delta_{m-1}^{j-1} \pmod{p^{\frac{1}{2}}}.$$

Comparing the coefficients, we have $T^{h(p)}\equiv -m\Delta_{m-1}^{m-1}\equiv -m! \pmod p$ and

$$(5)\quad h(p)T^{h(p)-1}U\equiv 1+\tfrac{1}{2}\sum_{i,j\geq 1,\ i+j=m}\Delta_{m-1}^{i-1}\Delta_{m-1}^{j-1} \pmod p.$$

By a lemma on finite differences:

$$\sum_{i,j\geq 1,\ i+j=m}\Delta_{m-1}^{i-1}\Delta_{m-1}^{j-1}=\frac{1}{(2m-1)\binom{2(m-1)}{m-1}}\Delta_{2m-1}^{m-1}-\frac{2(m-1)!}{m}B_m,$$

and $T^2\equiv -1 \pmod p$, the author obtains $h(p)U\equiv TB_{\frac{1}{2}(p-1)}$. He also announces an analogous congruence about the class number of $R(\sqrt{pl})$, modulo p, but no proof is given.

L.-K. Hua (Urbana, Ill.).

Hasse, Helmut. Die Einheitengruppe in einem total-reellen nicht-zyklischen kubischen Zahlkörper und im zugehörigen bikubischen Normalkörper. Arch. Math. 1, 42–46 (1948).

If Γ is a normal field over the field of rational numbers and if $\mathfrak{G}$ is the corresponding Galois group, the group of units of Γ can be considered as an operator-group H which possesses the elements of $\mathfrak{G}$ as operators. The author announces results which disclose the structure of the operator-group H completely in the following cases: (1) Γ a real cyclic cubic field; (2) Γ a real cyclic biquadratic field; (3) Γ the normal field belonging to a totally real noncyclic cubic field K. For instance, in the first case, H is cyclic as an operator-group and the generating element is uniquely determined apart from trivial substitutions. In the second case, the corresponding statement is true for the subgroup H_0 of those units whose relative norm with regard to the quadratic subfield is ± 1. The possible types of groups H can be characterized as extensions of H_0. Corresponding theorems are given in the third case too. As an application results concerning the units in K are obtained.

R. Brauer (Ann Arbor, Mich.).

Karpelevič, F. I. Pseudonorms in the ring of integers. Uspehi Matem. Nauk (N.S.) 3, no. 5(27), 174–177 (1948). (Russian)

Let ϕ be a real-valued function on the integers satisfying $\phi(0)=0$, $\phi>0$ otherwise, and the triangle inequalities for addition and multiplication. The author proves that the induced topology can be described by taking as neighborhoods of 0 the ideals of a geometric progression. As the author acknowledges, a more general result was proved by Mahler for rings of algebraic integers [Acta Math. 67, 283–328 (1936)]. The proof follows the same lines as Mahler's, but takes advantage of the simplifications possible in this special case.

I. Kaplansky (Princeton, N. J.).

ANALYSIS

Popoviciu, Tiberiu. Sur une inégalité. Mathematica, Timişoara 23, 127–128 (1948).

Given a continuous strictly increasing real function φ with inverse φ^{-1} on an open real interval I and a sequence $M=(x_\mu)$ of m points x_μ of I, define $m_\varphi M=\varphi^{-1}(\bar{\varphi}(M))$, where $\bar{\varphi}(M)=m^{-1}\sum\varphi(x_\mu)$ is the arithmetic mean of φ over M. Furthermore, given two real functions φ and ψ on I and a sequence P of mn points of I, define $m_\varphi n_\psi P=m_\varphi(n_\psi(N_\mu))$, where $P=(N_\mu)$ is the decomposition of P into m consecutive sequences N_μ each containing n points. Let P' be a monotone rearrangement of P. It is shown that $m_\varphi n_\psi P'\geqq m_\varphi n_\psi P$ provided the function $\varphi\psi^{-1}$ is convex. This generalizes a known result of L. A. Le Cointe which corresponds to the special case $m=2$, $\varphi=x$, $\psi=\log x$ [Nouv. Ann. Math. (1) 2, 372–374 (1843)].

W. Gustin (Bloomington, Ind.).

Horvath, J. Note sur un problème de L. Fejér. Bull. École Polytech. Jassy [Bul. Politehn. Gh. Asachi. Iaşi] 3, 164–168 (1948).

The author studies the problem of determining conditions on a mean-value function $\mu(y_1, y_2)$ which insure the existence of a continuous function $f(x)$, $a\leqq x\leqq b$, which satisfies (1) $f\{\frac{1}{2}(x_1+x_2)\}=\mu[f(x_1), f(x_2)]$ for all x_1, x_2 with $a\leqq x_1<x_2\leqq b$, and which assumes given boundary values $f(a)=A$, $f(b)=B$. By the theorem of Kolmogoroff and Nagumo, $\mu(y_1, y_2)$ can be expressed as a function of the form $\mu(y_1, y_2)=\varphi^{-1}[\frac{1}{2}\{\varphi(y_1)+\varphi(y_2)\}]$, where φ is continuous and strictly increasing, if and only if $\mu(y_1, y_2)$ is symmetric, reflexive, monotone, continuous, and bisymmetric. It follows that, if $\mu(y_1, y_2)$ satisfies these conditions, then a solution of (1) is given by

$$f(x)=\varphi^{-1}[\varphi(A)(b-x)/(b-a)+\varphi(B)(x-a)/(b-a)].$$

Conversely, if $f(x)$ is a solution of (1), then $\varphi[f(x)]$ is a linear function of x.

If $M(F; x_0, y_0; r)$ denotes a similarly restricted mean-value function of F on a circle with center (x_0, y_0) and radius r, the author shows similarly, by means of a theorem of Jessen, that the equation $F(x_0, y_0)=M(F; x_0, y_0; r)$ has a unique solution taking on given continuous boundary values. By analogy with the above one-dimensional case, the function $\varphi[F(x, y)]$ is harmonic. It is shown finally that if $\varphi(x)$ is concave then $f(x)$ is convex and $F(x, y)$ is subharmonic.

E. F. Beckenbach (Los Angeles, Calif.).

Aczél, J., und Fenyö, St. Über die Theorie der Mittelwerte. Acta Univ. Szeged. Sect. Sci. Math. 11, 239–245 (1948).

The authors determine conditions on a mean-value function $\mu(x_1, x_2)$ which are necessary and sufficient for the existence of a strictly monotone function $f(x)$ having a continuous second derivative, and a constant p, $0<p<1$, such that $f[\mu(x_1, x_2)]=pf(x_1)+(1-p)f(x_2)$ for all x_1, x_2. [Cf. the preceding review.] These conditions are given in such a form that the function $f(x)$ can be exhibited explicitly as an integral. Thus one of the conditions can be written as

$$\frac{\partial\mu}{\partial x}\frac{\partial\mu}{\partial y}=\sigma(x)/\tau(y);$$

then $f(x)=c\int_a^x\sigma(s)ds=k\int_b^x\tau(s)ds$, where the constants a, b, c, and k are arbitrary. For example, if μ is the weighted arithmetic, geometric, harmonic, or power mean, then $f(x)$ is a linear function of x, $\log x$, x^{-1}, or x^t, respectively.

E. F. Beckenbach (Los Angeles, Calif.).

Shniad, Harold. On the convexity of mean value functions. Bull. Amer. Math. Soc. **54**, 770–776 (1948).

The mean value function M of n positive a's weighted by n positive ξ's with $\sum\xi=1$ is a continuous nondecreasing function of t for all real t including $t=\pm\infty$. Except for $t=0, \pm\infty$ this function is given by $M=(\sum\xi a^t)^{1/t}$. It is shown that M and $\log M$ as functions of t for fixed a and ξ are convex near $t=-\infty$ and concave near $t=+\infty$. Furthermore, although M and $\log M$ are convex-concave functions of t for $n=2$, they need not be so, as illustrated by example, for $n>2$.

W. Gustin (Bloomington, Ind.).

Calculus

***Severi, Francesco, and Scorza Dragoni, Giuseppe. Lezioni di Analisi.** Vol. 2, part 1. Nicola Zanichelli, Bologna, 1942. vii+398 pp.

[The second edition of vol. 1 appeared in 1938.] The topics treated in the present volume are: series of functions of real and complex variables; geometrical applications of the differential calculus; definite and indefinite integrals; methods of integration; partial differentiation; curvilinear and multiple integrals. The book is intended primarily for engineering students and the body of the text gives a fairly elementary but rigorous treatment of the customary subject matter under the headings mentioned above. But every chapter ends with complements and (a few) exercises which together occupy two fifths of the text and which give the authors a chance of opening vistas into more advanced fields. Here the reader finds an account of analytic functions of one and several variables, the Lebesgue theory of measure and integration, the area problem, differential geometry, tensor calculus, functions of bounded variation, Stieltjes integrals and many other topics. An abundance of dates and historical references enhance the value of this excellent treatise.

E. Hille (New Haven, Conn.).

Castoldi, L. Sopra la determinazione del grado di dipendenza lineare tra funzioni di più variabili indipendenti. I. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **3**, 498–502 (1947).

It is stated but not proved that a necessary and sufficient condition for the linear dependence of n analytic functions $y_1, \cdots, y_n$ of m independent variables $x_1, \cdots, x_m$, and the existence of $h>0$ linearly independent homogeneous relations with constant coefficients among the y's, is that $n-h$ be the rank of the following matrix of n columns and infinitely many rows. The first row is $y_1\ y_2\ \cdots\ y_n$. Each later row consists of partial derivatives of the y's, with the same partial derivative of each y_i. A row of partial derivatives of a given order precedes all rows of derivatives of higher orders, but otherwise the arrangement of rows is arbitrary. The error in the proof consists in the assumption, in connection with equations (9), page 501, that the determinant of the y's and their partial derivatives with respect to any particular one of the x's, through order $n-h'-1$, is not zero; following merely the hypothesis that the rank of the matrix mentioned above is $n-h'$.

A. B. Brown.

Castoldi, L. Sopra la determinazione del grado di dipendenza lineare tra funzioni di più variabili indipendenti. II. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **4**, 46–49 (1948).

It is shown how the criterion of note I [see the preceding review] can be applied in a finite number of steps. In particular, if all the rows of derivatives of a given order do not increase the rank when added to the matrix containing only the derivatives of lower orders, then the maximum rank has been reached. The following false theorem is stated as a special case. If $m=n-1$, a necessary and sufficient condition for the linear independence of $y_1, \cdots, y_n$ is that the determinant of the y's and their first partial derivatives be not zero. That the condition is not necessary is shown by the example of the following three functions of two variables: (x_1+x_2), $(x_1+x_2)^2$, $(x_1+x_2)^3$, which are linearly independent although the determinant in question is identically zero.

A. B. Brown (Flushing, N. Y.).

Castoldi, L. Sopra la determinazione del grado di dipendenza lineare tra funzioni di più variabili indipendenti. III. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **4**, 190–192 (1948).

[Cf. the preceding review.] The principal theorem here is the following, which is false. There can be no criterion of sufficiency for linear dependence of nonanalytic functions, which involves only a finite number of derivatives. A counter-example is furnished by the following well-known theorem, whose proof is obtained by solving a simple differential equation. If y_1 and y_2 are functions of x not zero on an interval of the x-axis on which y_1' and y_2' exist and are continuous, and if the determinant of the y's and their first derivatives is identically zero on the interval, then y_1 and y_2 are linearly dependent.

A. B. Brown (Flushing, N. Y.).

Castoldi, L. Sopra la determinazione del grado di dipendenza lineare tra funzioni di più variabili indipendenti. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **4**, 533 (1948).

A correction to the author's note III of the same title, reviewed above.

Theory of Sets, Theory of Functions of Real Variables

Hall, Marshall, Jr. Distinct representatives of subsets. Bull. Amer. Math. Soc. **54**, 922–926 (1948).

It is shown that there is a system of distinct representatives of an indexed infinite system of sets, each of which is a subset of a given set, if each set of the system is finite, and for every finite k, every k distinct sets contain together at least k distinct elements. This extends P. Hall's theorem [J. London Math. Soc. **10**, 26–30 (1935)]. A representative of a set belongs to the set, and it is emphasized that the sets are distinguished only by their indices so that distinct sets may contain the same elements. Also a lower bound for the number of systems of distinct representatives is given which leads to the following: there are at least $n!\,(n-1)!\cdots 2!\,1!$ different n by n Latin squares.

J. Riordan (New York, N. Y.).

Sudan, Gabriel. Sur les nombres delta. Acad. Roum. Bull. Sect. Sci. **26**, 212–223 (1946).

According to G. Hessenberg [Grundbegriffe der Mengenlehre, Vandenhoeck und Ruprecht, Göttingen, 1906, § 82],

a transfinite ordinal number is called a δ-number if $\alpha \cdot \delta = \delta$ for every ordinal number $\alpha (\neq 0)$ smaller than δ. The author's main result is the following theorem. Every transfinite ordinal number β has the same power as the greatest δ-number contained in β and as the smallest δ-number greater than β. This theorem corresponds to an analogous theorem of Hessenberg [loc. cit., § 66] about the principal numbers. The author proves his result without using the general formulas concerning the alephs, and he derives some of these formulas from his theorem. Moreover, he gives new simple proofs of results of Hessenberg [loc. cit.] to the effect that every initial number is a δ-number and an ϵ-number.

A. Rosenthal (Lafayette, Ind.).

Sudan, Gabriel. Sur la "forme normale" de Cantor et la définition de Hessenberg pour la puissance des nombres ordinaux. Acad. Roum. Bull. Sect. Sci. **27**, 108–117 (1947).

G. Hessenberg [Jber. Deutsch. Math. Verein. **16**, 130–137 (1907)] introduced a constructive definition of the power of an ordinal number. By means of this definition the author gives a direct proof of Cantor's representation of the ordinal numbers by normal forms. Moreover, he shows that this representation and Hessenberg's definition are equivalent. *A. Rosenthal* (Lafayette, Ind.).

Sudan, Gabriel. Sur une propriété des nombres epsilon. Acad. Roum. Bull. Sect. Sci. **27**, 258–264 (1947).

The following theorem has been proved by G. Hessenberg [Jber. Deutsch. Math. Verein. **16**, 130–137 (1907)] and E. Jacobsthal [Math. Ann. **66**, 145–194 (1908)]. If for one ordinal number α with $1<\alpha<\beta$ the relation $\alpha^\beta=\beta$ holds, then for every number γ with $1<\gamma<\beta$ the relation $\gamma^\beta=\beta$ also holds. This then means that β is an ϵ-number. The author gives a new proof of this theorem and derives from it the following two consequences. (1) If γ is a transfinite ordinal number and $\alpha>1$, $\beta>1$ are two ordinal numbers satisfying the relation $(\alpha^\beta)^\gamma=\alpha^{(\beta\gamma)}$, then γ is an ϵ-number. (2) If γ is a limit number and if there exist two ordinal numbers α, β with $1<\alpha<\gamma$, $1<\beta$, such that $\alpha\gamma=\beta^\gamma$, then γ is an ϵ-number. *A. Rosenthal* (Lafayette, Ind.).

Popoviciu, Tiberiu. Sur la formule des accroissements finis. Mathematica, Timişoara **23**, 123–126 (1948).

Let n be a positive integer and let $x_0=a<b=x_{n+1}$. A sequence $(x_1, \cdots, x_n)$ is called increasing if $x_0\leqq x_1\leqq\cdots\leqq x_n\leqq x_{n+1}$, strictly increasing if $x_0<x_1<\cdots<x_n<x_{n+1}$. Given a function f on the closed interval $a\leqq x\leqq b$, define for each increasing sequence $(x_1, \cdots, x_n)$ the function f_n as the trapezoidal area decomposition

$$f_n(x_1, \cdots, x_n)=\sum_{k=0}^{n}\tfrac{1}{2}[f(x_{k+1})-f(x_k)]\cdot(x_{k+1}-x_k).$$

The following generalization of the mean value theorem is proved. If f is continuous in the closed interval $a\leqq x\leqq b$ and differentiable in the open interval $a<x<b$, then a strictly increasing sequence $(x_1, \cdots, x_n)$ exists for which f_n attains an absolute extreme value; furthermore, for any such sequence $f(x_k)=[f(x_{k+1})-f(x_{k-1})]/(x_{k+1}-x_{k-1})$.

W. Gustin (Bloomington, Ind.).

Matos Peixoto, Mauricio. On the existence of derivatives of generalized convex functions. Summa Brasil. Math. **2**, no. 3, 35–42 (1948).

Let S be the open vertical strip erected on an open interval I. Consider a family $\mathfrak{F}$ of functions, called $\mathfrak{F}$-linear functions, defined on I and having the property that the graph of exactly one function of $\mathfrak{F}$ passes through two given points of S with different abscissas. A function f on I is called $\mathfrak{F}$-convex, provided that for every two distinct points a and b of I the $\mathfrak{F}$-linear function F passing through the points $(a, f(a))$ and $(b, f(b))$ of S satisfies the inequality $f(x)\leqq F(x)$ for all x between a and b. Several properties of $\mathfrak{F}$-linear and $\mathfrak{F}$-convex functions have been established if the F-linear functions are continuous [Beckenbach, Bull. Amer. Math. Soc. **43**, 363–371 (1937)]. The author shows that if the $\mathfrak{F}$-linear functions are differentiable almost everywhere, then any $\mathfrak{F}$-convex function is differentiable almost everywhere. Furthermore, if the second derivatives of the $\mathfrak{F}$-linear functions exist everywhere and are in a certain sense locally uniformly bounded, then any $\mathfrak{F}$-convex function is differentiable except for a countable set.

W. Gustin (Bloomington, Ind.).

Frumkin, P. B. On a theorem of D. F. Egorov on measurable functions. Doklady Akad. Nauk SSSR (N.S.) **60**, 973–975 (1948). (Russian)

G. Tolstov has shown [C. R. (Doklady) Acad. Sci. URSS (N.S.) **22**, 305–307 (1939)] that Egoroff's theorem does not hold for a continuous parameter. Now the author proves the following generalization of the theorem. If $f(t, s)$, $0\leqq t, s\leqq 1$, is for every t measurable and almost everywhere finite, and if $\lim_{t\to t_0} f(t, s)=f(t_0, s)$ for almost all s, then to every $\epsilon, \delta>0$ there exists a set E and a positive number φ such that $m(E)>1-\delta$ and $|t-t_0|<\varphi$ imply that ess bd $|f(t,s)-f(t_0,s)|$, for $s\in E$, is less than ϵ. Formally this theorem differs from Egoroff's theorem by the use of the essential bound in place of the bound. The proof uses Kantorovich's semi-ordered spaces and the following lemma. If (i) $f(t, s)\in L_s^p(0, 1)$, $p\geqq 1$, for every t in $(0, 1)$, (ii) $\lim_{t_n\to t_0} f(t_n, s)=f(t_0, s)$ almost everywhere, (iii) $\sup_t f(t, s)\in L^p$, then for arbitrary $\epsilon, \delta>0$ there exist a set E and a $\varphi>0$ such that ess bd $|f(t,s)-f(t_0,s)|<\epsilon$ $(s\in E)$ for all $t\in(t_0-\varphi, t_0+\varphi)$. *František Wolf.*

Stone, M. H. Notes on integration. III. Proc. Nat. Acad. Sci. U. S. A. **34**, 483–490 (1948).

[Parts I and II appeared in the same vol., 336–342, 447–455 (1948); these Rev. **10**, 24, 107.] The author proves Fubini's theorem for his axiomatic theory of integration. As applications he discusses (1) an analogue of Ambrose's theorem on the relations among measures in a locally compact group, a closed subgroup, and the induced coset space, and (2) Jessen's theory of integration in infinite product spaces. *P. R. Halmos* (Chicago, Ill.).

Andersen, Erik Sparre, and Jessen, Børge. On the introduction of measures in infinite product sets. Danske Vid. Selsk. Mat.-Fys. Medd. **25**, no. 4, 8 pp. (1948).

Doob [Trans. Amer. Math. Soc. **44**, 87–150 (1938)] stated that a probability measure defined on the finite dimensional measurable cylinders of an infinite dimensional product space may be extended to a measure on the class of all measurable sets. This result has been known for some time in case the factor spaces are intervals and also in case the given measure is determined multiplicatively by its one dimensional sections. The authors construct an example to show that in general the proposition is not true.

P. R. Halmos (Chicago, Ill.).

Henstock, R. On interval functions and their integrals. II. J. London Math. Soc. **23**, 118–128 (1948).

C'est un deuxième mémoire de l'auteur sur l'intégrale de Burkill d'une fonction d'intervalle non nécessairement addi-

tive [cf. même J. 21, 204–209 (1946); ces Rev. 8, 572]. (1) La bipartition de l'intervalle d'intégration est étudiée et une égalité de S. Saks est infirmée par un contre-exemple. Figurent ensuite des inégalités relatives à l'approximation de l'intégrale par des sommes, et une étude des discontinuités. (2) Relations entre la notion de k-intégration précédemment introduite par l'auteur (les points singuliers devenant points de division à partir d'un certain rang), et la notion de σ-limite de Getchell-Hildebrandt. (3) Étude de l'intégration de Burkill et de la k-intégration des fonctions d'intervalle à variation bornée. (4) Fonctions d'intervalle monotones vis-à-vis d'une subdivision, c'est-à-dire par exemple telles que $g(a,c)\geqq g(a,b)+g(b,c)$ $(a<b<c)$. Les résultats de Burkill sont complétés. (5) Pour terminer, l'auteur ajoute des précisions à certains résultats du premier mémoire: il intervient ici un ensemble, ses recouvrements par des ensembles d'intervalles, et deux fonctions d'intervalles. *R. de Possel* (Alger).

Viola, Tullio. Sui fondamenti della teoria della misura in uno spazio cartesiano a tre o più dimensioni. Period. Mat. (4) **26**, 32–44 (1948).

The author generalizes from three to an arbitrary dimension n a result of Picone [Period. Mat. (4) **25**, 181–195 (1947); these Rev. **9**, 339] on the volume of elementary configurations in Euclidean space. The figures discussed are right parallelopipeds and polyhedra. *H. H. Goldstine.*

Tolstov, G. P. On the total differential. Uspehi Matem. Nauk (N.S.) **3**, no. 5(27), 167–170 (1948). (Russian)

The usual proof of Cauchy's theorem is adapted to show that, for $Pdx+Qdy$ to be a total differential in a simply connected region G, it suffices that in this region P and Q have total differentials and $\partial P/\partial y=\partial Q/\partial x$.

M. M. Day (Princeton, N. J.).

Müller, M. Folgen gleichgradig integrierbarer Funktionen und die Vertauschbarkeit von Grenzübergang und Integration. Math. Z. **51**, 294–305 (1948).

Using the obvious definition of uniform Riemann integrability, the author shows that that condition implies the term by term Riemann integrability of a convergent sequence and that several other conditions, e.g., equicontinuity, imply uniform Riemann integrability. The proofs are completely elementary. *P. R. Halmos* (Chicago, Ill.).

Conti, R. Estensione alle successioni di funzioni a variazione limitata di un criterio di Pólya-Cantelli per la convergenza uniforme su intervalli infiniti. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **4**, 61–65 (1948).

Let $f_n(x)$ be defined on (a,∞) and let $v_n(x)$ be the total variation of $f_n(x)$. If the $v_n(x)$ are uniformly bounded and $f_n(x)\to f_0(x)$ then the convergence is shown to be uniform provided (1) $f_n(x\pm)\to f_0(x\pm)$, (2) $f_n(\infty)=f_0(\infty)$, (3) $v_n(\infty)\to v_0(\infty)$. *W. Feller* (Ithaca, N. Y.).

Cesari, L. Sull'area secondo Lebesgue delle superficie continue. I. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **3**, 486–489 (1947).

Cesari, L. Sull'area secondo Lebesgue delle superficie continue. II. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **3**, 489–495 (1947).

If a sequence of surfaces converges uniformly and in Lebesgue area, so does the sequence of their projections into any plane. *H. Federer* (Providence, R. I.).

Cesari, L. Sopra un teorema di approssimazione per le superficie continue in forma parametrica. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **4**, 33–39 (1948).

Some approximation theorems connected with the author's paper in Ann. Scuola Norm. Super. Pisa (2) **13** (1944), 77–117 (1948); these Rev. **9**, 505. *H. Federer.*

Dolcher, Mario. Geometria delle trasformazioni continue. I. Sopra un teorema di Radó. Ann. Scuola Norm. Super. Pisa (2) **14** (1945), 99–116 (1948).

Let R be a bounded, simply connected Jordan region in a Euclidean plane π. Let C be the positively oriented boundary curve of R. If Φ denotes a continuous mapping from R into a Euclidean plane π', then for every point Q' of π' one can define, in a well-known manner, a topological index $O[Q',\Phi(c)]$, for every oriented Jordan curve c in R. As a generalization of a theorem of the reviewer, the author proves the following theorem. Let G be the inverse set, under Φ, of a point Q_0', and suppose that there exists a component G_0 of G with the following property: $O[Q_0',\Phi(c)]\cong O[Q_0',\Phi(C)]$ for every Jordan curve c in R that contains G_0 in its interior and does not intersect G. Let $K(Q_0')$ be the component of the set $\pi'-\Phi(C)$ that contains Q_0'. Then every point of $K(Q_0')$ has at least $O[Q_0',\Phi(C)]$ inverse points under Φ. The author points out relationships between this result and similar results due to the reviewer and to L. Cesari. *T. Radó.*

Theory of Functions of Complex Variables

Heffter, Lothar. Kurvenintegrale und Begründung der Funktionentheorie. Arch. Math. **1**, 77–79 (1948).

The author indicates how analytic functions $f(z)$, $z=x+iy$, may be defined without using the notion of a derivative. The author uses the Cauchy and Morera theorems for oriented rectangles to characterize analytic $f(z)$, thus avoiding the machinery of line integrals. *M. Reade.*

Zevi, Maria. Sopra una proprietà di limite caratteristica delle funzioni olomorfe. Pont. Acad. Sci. Acta **5**, 143–152 (1941).

For a function $f(z)$ defined in a domain D, and for points $z_1,\cdots,z_{n+1}$ in D, let $\varphi(z_1,\cdots,z_{n+1})$ be defined by

$$\varphi(z_1,\cdots,z_{n+1})\begin{vmatrix}1 & z_1 & \cdots & z_1^n\\ & \cdots & \cdots & \\ 1 & z_{n+1} & \cdots & z_{n+1}^n\end{vmatrix}=\begin{vmatrix}1 & z_1 & \cdots & z_1^{n-1} & f(z_1)\\ & \cdots & \cdots & & \\ 1 & z_{n+1} & \cdots & z_{n+1}^{n-1} & f(z_{n+1})\end{vmatrix}.$$

If $f(z)$ is holomorphic in D, then for z_0 in D we have

$$\text{(I)}\qquad \lim_{(z_j)\to z_0}\varphi(z_1,\cdots,z_{n+1})=f^{(n)}(z_0)/n!.$$

For $n=2$, Picone [Period. Mat. (4) **21**, 141–150 (1941); these Rev. **3**, 145] has investigated the question of determining the class of functions for which the limit in (I) exists and is finite, and has obtained two types of results according as the points $z_1,\cdots,z_{n+1}$ are restricted to be collinear with z_0 or are not so restricted.

The author now obtains the following similar results for general n. (1) Let $f(z)=f(x+iy)$ have continuous partial derivatives of the first n orders with respect to x and y in D, and let $t_1,\cdots,t_{n+1}$ be distinct fixed real numbers different from 0. Then for each point z of D, $\lim_{\zeta\to 0}\varphi(z+t_1\zeta,\cdots,z+t_{n+1}\zeta)$, with ζ restricted to $n+1$ directions of approach to 0, exists

and is finite if and only if $f(z)$ is the sum of a holomorphic function of z and a polynomial of degree at most $n-1$ in x and y. (2) Let $f(z)=f(x+iy)$ be differentiable in the sense of Stolz in D, and let ω be a primitive $(n+1)$th root of unity, with $n\geqq 2$. Then for each point z of D, $\lim_{\zeta\to 0}\varphi(z+\zeta, \cdots, z+\epsilon^n\zeta)$, with ζ restricted to a single direction of approach to 0, exists and is finite if and only if $f(z)$ is a holomorphic function of z. [Cf. also Picone, same vol., 155–158 (1941); these Rev. 7, 149.] *E. F. Beckenbach* (Los Angeles, Calif.).

Capyrin, V. N. On the problem of Hurwitz for transcendental equations. Akad. Nauk SSSR. Prikl. Mat. Meh. 12, 301–328 (1948). (Russian)

The author gives a detailed solution of Hurwitz's problem for the following particular equation:

$$f(z)=(a_1z^2+a_2z+a_3)\cosh z+(b_1z^2+b_2z+b_3)\sinh z=0,$$

where the a's and the b's are real; that is to say he gives conditions bearing on these coefficients in order that all the zeros of $f(z)=0$ have negative real parts. This generalizes a theorem ascribed to Chebotareff which solves the problem when the coefficients are positive. The conditions are expressed in forms too involved to be stated in a few sentences. *S. Mandelbrojt* (Paris).

Goluzin, G. M. Some covering theorems in the theory of analytic functions. Mat. Sbornik N.S. 22(64), 353–372 (1948). (Russian)

In the first part of the paper special analytic functions are constructed which are connected with the mapping of the unit circle onto the interior of circular polygons. Similar functions were used to obtain covering theorems by Carathéodory [C. R. Acad. Sci. Paris 144, 1203–1206 (1907)]. In the second part, the author considers functions $w=f(z)=z^p(c_0+c_1z+\cdots)$, p a positive integer, which are regular or meromorphic in $|z|<1$, and he obtains precise bounds for $|c_0|$ under the supposition that $f(z)$ omits certain values in $|z|<1$ (for example, the values $ce^{2\pi ik/n}$, $k=1, 2, \cdots, n$, $n\geqq 3$). Under additional restrictions on $f(z)$ theorems are obtained concerning the covering of circles with centers at $w=0$. In the final section of the paper the author applies the method of Grötzsch to functions of the above form which are p-valent in $|z|<1$. This section is a generalization to p-valent functions of the method applied to schlicht functions in chapter IV of the author's article [Uspehi Matem. Nauk 6, 26–89 (1939); translation, Office of Naval Research, 1947; these Rev. 1, 49; 8, 575]. *D. C. Spencer* (Stanford University, Calif.).

Kufarev, P. P. A solution of the boundary problem for an oil well in a circle. Doklady Akad. Nauk SSSR (N.S.) 60, 1333–1334 (1948). (Russian)

The author considers the flow of oil in a unit circle in the presence of a single well inside the circle. The analysis is based on the following theorem. Let $\alpha(t)$, $\beta(t)$, $A_k(t)$, $a_k(t)$, $|t|<t_0$, be the solutions of the system of equations

$$\beta\bar{\beta}-\sum_{k=1}^{n}\bar{A}_k\beta/\bar{a}_k{}^2=D_1-2t,\quad \alpha-\sum_{k=1}^{n}A_k/a_k=D_2,$$

$$\alpha+\beta/\bar{a}_k-\sum_{s=1}^{n}A_s\bar{a}_k/(a_s\bar{a}_k-1)=C_k,$$

$$\bar{A}_k\beta/\bar{a}_k{}^2-\sum_{s=1}^{n}\bar{A}_kA_s/(\bar{a}_ka_s-1)^2=C_k',$$

where D_1, D_2, C_k, C_k' are constants, and $|a_k|>1$; then the function

$$z=\beta W+\alpha+\sum_{k=1}^{n}A_k/(W-a_k)$$

satisfies the boundary condition

$$\omega^{-1}z_t'(\omega, t)\bar{z}_t'(\omega^{-1}, t)+\omega\bar{z}_t'(\omega^{-1}, t)z_w'(\omega, t)=-2,\qquad \omega=e^{i\theta}.$$

A. W. Boldyreff (Albuquerque, N. M.).

Matildi, Pietro. Sulla rappresentazione conforme di domini appartenenti a superficie di Riemann su domini di un tipo canonico assegnato. Ann. Scuola Norm. Super. Pisa (2) 14 (1945), 81–90 (1948).

Démonstration de la proposition suivante. Un domaine A d'ordre de connexion $2\sigma+1$, placé sur une surface de Riemann et limité par un seul contour, peut être représenté conformément et biunivoquement sur un cercle multiplement connexe constitué par λ feuillets plans reliés par des lignes de passage convenables. L'auteur s'appuie sur les travaux de Cecioni [Ann. Univ. Toscane (N.S.) 12, 27–88 (1929)] et de Mme. Li Chiavi [Rend. Sem. Mat. Univ. Padova 3, 95–127 (1932)]. Il observe que, dans un ordre d'idées voisin, un travail de Cecioni [Rend. Circ. Mat. Palermo 25, 1–19 (1908)] a précédé celui de Hilbert [Nachr. Ges. Wiss. Göttingen 1909, 314–323]. L'entier λ qui figure dans l'énoncé est au moins égal à $n(2n-3)$, n étant le degré d'une courbe algébrique correspondant à A dans la théorie de Cecioni. Les intégrales abéliennes de première espèce sont utilisées. *G. Valiron* (Paris).

Wittich, H. Zum Beweis eines Satzes über quasikonforme Abbildungen. Math. Z. 51, 278–288 (1948).

Let $D_{z|w}$ denote the coefficient of dilatation in a quasiconformal mapping of the z-plane onto the w-plane. The condition

$$\iint_{|z|>r_0}(D_{z|w}-1)\,|z|^{-3}dxdy<\infty$$

implies $\lim_{z\to\infty}|w/z|=a$, $0<a<\infty$. This was proved by Teichmüller [Deutsche Math. 3, 621–678 (1938)] and the purpose of the present paper is to give a new proof based on a method of the reviewer [Acta Math. 58, 375–406 (1932)]. Applications are made to the problem [Ahlfors, Wolff] of finding conditions for the existence of an angular derivative in conformal mapping. The use of quasi-conformal mappings is very natural in this connection and considerable simplifications are achieved. *L. Ahlfors.*

Turán, P. On some examples in the theory of power series. Bull. Amer. Math. Soc. 54, 932–936 (1948).

Many of the power series examples in the literature, which have been constructed to show the independence of certain properties, have required either the use of van der Corput's estimations or of special number-theoretical functions. The author illustrates an elementary method for the construction of such examples by exhibiting a power series $\sum a_nz^n$ which converges uniformly in $|z|\leqq 1$ and is therefore continuous and bounded there, but for which the series $\sum|a_n|^{2-\epsilon}$ diverges. This example is

$$\sum_{n=4}^{\infty}n^{-\delta}\sum_{n^2+1}^{(n+1)^2}z^r\exp(-it\theta_n),$$

where $\delta=1+\epsilon/3$, and where $\theta_n=(n-2^r)/2^{r-1}$ for any r and for $n=2^r, 1+2^r, \cdots, 2^{r+1}-1$. *R. C. Buck.*

Boas, R. P., Jr. A class of gap theorems. Duke Math. J. 15, 725–728 (1948).

(I) Let β, γ be positive integers and let $\{n_p\}$, $\{m_p\}$ be strictly increasing sequences of positive integers. If $|a(n\beta+\gamma)| \leqq M(n \geqq 1)$, $\liminf_{p\to\infty} |a(n_p\beta+\gamma)| > 0$, $\lim_{p\to\infty} a(n_p\beta+\gamma+l)=0$ for $l=1, 2, \cdots, m_p$, then for no interval $|\theta-\theta_0|<\delta$ with $\delta > 2\alpha = 2\pi(1-\beta^{-1})$ can we have $|f(re^{i\theta})| \leqq \psi(\theta)$ where $f(z)=\sum a(n)z^n$, $\psi(\theta)$ being an integrable function ($0 \leqq r < 1$). (II) With the same conditions $f(z)$ has a singularity in every arc of $|z|=1$ of length greater than 2α provided that one of the following sets of conditions is satisfied:

(a) $|a(n\beta+\gamma)| = O(n^k)$, $\liminf |a(n_p\beta+\gamma)| n_p^{-k} \neq 0$, $\lim a(n_p\beta+\gamma+l)(n_p\beta+\gamma+l)^{-k}=0$ $(1 \leqq l \leqq m_p;\ k>0)$;

(b) $|a(n\beta+\gamma)| = O(e^{bn^\sigma})$, $\liminf |a(n_p\beta+\gamma)| e^{-bn_p^\sigma} \neq 0$, $\lim a(n_p\beta+\gamma+l)e^{-b(n_p\beta+\gamma+l)^\sigma}=0$ $(1 \leqq l \leqq m_p,\ 0<\sigma<1,\ b>0)$.

The result (I) contains a result of Ilieff [Annuaire [Godišnik] Univ. Sofia. Fac. Phy.-Math. Livre 1. 42, 67–81 (1946); these Rev. 9, 21]. The author points out that while his paper was in the course of publication, more general results were announced by S. Agmon [C. R. Acad. Sci. Paris 226, 1673–1674 (1948); these Rev. 9, 576]. *S. Mandelbrojt.*

Boas, R. P., Jr. Exponential transforms and Appell polynomials. Proc. Nat. Acad. Sci. U. S. A. 34, 481–483 (1948).

An integral of the form $\int_0^\infty f(t)e^{\alpha z t}dt$ ($|\alpha|=1$) is termed an exponential transform. Every function analytic in a bounded convex n-gon is shown to be the sum of n transforms with non-collinear α's; in particular, every analytic function is in infinitely many ways the sum of three transforms. The proof stems from Cauchy's formula. This transform representation for analytic functions has application to the problem of expanding functions in series of Appell polynomials, thus (among other things) obtaining by a new method results of the reviewer [Bull. Amer. Math. Soc. 47, 885–898 (1941); these Rev. 3, 111]. It is remarked that certain cases of series of polynomial sets of "type zero" could also be handled. Details are promised elsewhere.

I. M. Sheffer (State College, Pa.).

Mergelyan, S. N. On best approximation in adjacent regions. Doklady Akad. Nauk SSSR (N.S.) 61, 981–983 (1948). (Russian)

In this and in the following two papers, some theorems and observations on best approximation by polynomials are stated without proof. Let D_1, D_2 be simply-connected regions the boundaries of which are the complete boundaries of the corresponding complementary regions, and set $D=D_1+D_2$. Let $f_i(z)$, $i=1, 2$, be regular in D_i and continuous in $\bar{D}_i$, and set $\rho(n)=\max |f_i(z)-P_n(z)|$ $(z \in \bar{D}_i,\ i=1, 2)$ for all polynomials P_n of degree n. Suppose D_1, D_2 have only one point ($z=0$) in common, and that $f_1(0)=f_2(0)$, and set $f(z)=f_i(z)$ for $z \in \bar{D}_i$. Let $d(R)$ be the distance from $z=0$ to the level line L_R ($R>1$) of the complement to $\bar{D}$.

(I) If D_1, D_2 are convex, and $\omega(\delta)$ is the modulus of continuity of $f^{(k)}(z)$ on $\bar{D}$, then

$$\rho(n) < c\{d(1+n^{-1}\log n)\}^k \omega\{d(1+n^{-1}\log n)\},$$

c = constant. Suppose the boundaries of D_1, D_2 have near $z=0$ the respective equations $y=\varphi(x)=\varphi(-x)$ and $y=-\varphi(x)$ (for $|x|<a$), where $\varphi(x)$ is monotonic decreasing to zero with $|x|$. If D_1, D_2 meet at an angle: $\varphi(x)=q|x|$, $q>0$, then from (I): (1) $\rho(n) < c(n^{-1}\log n)^{kQ}\omega\{(n^{-1}\log n)^Q\}$, c = constant, $Q=1-2\pi^{-1}\tan^{-1} q$; and if D_1, D_2 have contact of algebraic order, i.e., $c_1x^m < \varphi(x) < c_2x^m$ $(m>1)$, then (2) $\rho(n) < c(\log n)^{-k/(m-1)}\omega\{(\log n)^{-k/(m-1)}\}$. (This estimate cannot be improved, and (1) is close to being sharp.)

Let the polynomials $\{P_n(z)\}$ converge uniformly in $\bar{D}_i$ to $f_i(z)$, and set $r_i(n)=\max |f_i(z)-P_n(z)|$ $(z \in \bar{D}_i)$. (II) If D_1, D_2 touch at $z=0$, and if

$$\liminf \{\log r_1(n)r_2(n)\}/\{\log d(1+1/n)\} = A,$$

then for every $\epsilon>0$, the derivative of $f(z)$ of order $[(A-\epsilon)/2]$ is continuous in $\bar{B}_1+\bar{B}_2$ and satisfies there a Lipschitz condition of order $(A-\epsilon)/2$. (B_i is any subregion of D_i with the property that the ratio of the distance of any point of $\bar{B}_i$ from $z=0$ to the distance of that point from the boundary of D_i is uniformly bounded.)

Let the polynomials $\{P_n(z)\}$ converge to $f(z)$ in $\bar{D}$, and set $r(n)=\max |f(z)-P_n(z)|$, $z \in \bar{D}$. (III) If

$$r(n) < e^{-\lambda_n}, \quad \lambda = \lambda_n = e^{s_n},\ s_n = \{d(1+1/n)\}^{-1},$$

then from the convergence of $\{P_n\}$ to zero in $\bar{D}_1$ it follows that $\{P_n\}$ likewise converges to zero in $\bar{D}_2$; so that the function $f_1(z)$ to which $\{P_n\}$ converges in $\bar{D}_1$ determines at the same time the function to which $\{P_n\}$ converges in $\bar{D}_2$, which is the "quasianalytic" continuation of $f_1(z)$ into D_2.

I. M. Sheffer (State College, Pa.).

Mergelyan, S. N. On best approximation in a complex region. Doklady Akad. Nauk SSSR (N.S.) 62, 23–26 (1948). (Russian)

Let D be a finite region, identical with the set of interior points of its closure $\bar{D}$, and let $f(z)$ be regular in D and continuous in $\bar{D}$. Set $\rho_n=\rho_n(D; f)=\max |f(z)-P_n(z)|$ $(z \in \bar{D})$ for all polynomials P_n of degree not exceeding n. The rate at which $\rho_n \to 0$ is examined under various assumptions concerning the boundary of D. If this boundary is an analytic curve it is known [W. E. Sewell, Degree of Approximation by Polynomials in the Complex Domain, Princeton University Press, 1942; these Rev. 4, 78] that (1) $\rho_n < cn^{-(k+\alpha)}$ (c = constant, k a positive integer, $0<\alpha<1$) is necessary and sufficient that $f(z)$ have a kth derivative satisfying in $\bar{D}$ a Lipschitz condition of order α. For nonanalytic regions results are less complete. Let $L(G; \alpha)$ be the class of functions regular in G and satisfying in $\bar{G}$ a Lipschitz condition of order α. (I) If D is bounded by a smooth curve, and $f(z)$ is regular in D, with $f^{(k)}(z) \in L(D; \alpha)$, then for arbitrary $\epsilon>0$, (2) $\rho_n < C_\epsilon \cdot n^{-(k+\alpha-\epsilon)}$ (C_ϵ independent of n). Inequality (2) is exact in that given any positive function $\varphi(\epsilon) \to \infty$ as $\epsilon \to 0$, there exists a region G with smooth boundary, and a function $F(z)$ regular in G, with $F^{(k)}(z) \in L(G; \alpha)$, such that $\rho_n(G; F) > \varphi(\epsilon)n^{-(k+\alpha-\epsilon)}$, $n > n(\epsilon)$. Conversely, if D has a smooth curve for boundary, and if $\rho_n < C \cdot n^{-(k+\alpha)}$, then, for arbitrary $\epsilon>0$, $f^{(k)}(z)$ satisfies a Lipschitz condition of order $\alpha-\epsilon$; and this result cannot be improved.

Let the boundary of D have the parametric equation $z=z(s)$, s = arc length, with $z'(s)$ continuous and of modulus of continuity $\gamma(\delta)$. (II) Let $f(z)$ be regular in D, with $f^{(k)}(z) \in L(D; \alpha)$. If (3) $\int_0^a x^{-1}\gamma(x)dx < \infty$ $(\alpha>0)$, then $\rho_n < c(n^{-1}\log n)^{k+\alpha}$; but if $\int_0^a x^{-1}\gamma(x)dx > |\log\log\epsilon| \cdot |\log\log\log\epsilon|$, then in general there exists a function $\varphi(z)$, with $\varphi^{(k)}(z) \in L(D; \alpha)$, yet $\limsup_{n\to\infty} \rho_n(D; \varphi)(n/\log n)^{k+\alpha} = \infty$. (III) Let (3) hold and also $\rho_n < cn^{-(k+\alpha)}$ (k a positive integer; $0<\alpha \leqq 1$). If $\alpha<1$, $f^{(k)}(z)$ satisfies in $\bar{D}$ a Lipschitz condition of order α; but if $\alpha=1$, a necessary and sufficient condition that $f^{(k)}(z)$ satisfy a Lipschitz condition of first order is that $\sum_{n=1}^\infty n^k\rho_n < \infty$.

Let the boundary of D have the polar coordinate equation $r=r(\varphi)$ $(=r(\varphi+2\pi))$, where r is single-valued and continuous, $0\leqq\varphi<2\pi$. Let all derived numbers of the function $\log r(\varphi)$ be uniformly bounded in modulus by m. (IV) If $f(z)$ is regular in D, and $\omega(\delta)$ is the modulus of continuity of $f^{(k)}(z)$ in D, then $\rho_n<c(n^{-1}\log n)^{(\pi k/\tau)\tan^{-1}m^{-1}}\omega\{(n^{-1}\log n)^{(1/\tau)\tan^{-1}m^{-1}}\}$. A particular case arises when the boundary has a corner point. Suppose two arcs of the boundary have the directions $\arg z=\pm\varphi/2$ at the point $z=0$, and that elsewhere the boundary has a continuously turning tangent. If $f^{(k)}(z)\varepsilon L(D;\alpha)$, then $\rho_n<C_\epsilon(1/n)^{\varphi(k+\alpha)/\pi-\epsilon}$; and this result is the best possible. Similarly if $z=0$ is a "re-entrant" point of contact of the two arcs; i.e., near $z=0$ the boundary has the equation $y=\pm\varphi(x)$ $(0<x<a)$ where $\varphi(x)$ and $\varphi'(x)$ decrease monotonically to zero as $x\to0$, and for some $\epsilon>0$ the interval $(-\epsilon,0)$ is contained in D. Suppose also that away from $z=0$ the boundary has a continuously turning tangent. (V) If $f(z)$ is regular in D, and the modulus of continuity of $f^{(k)}(z)$ does not exceed $\omega(\delta)$ on any closed subset of D, then $\rho_n<c[d(n^{-1}\log n)]^k\cdot\omega(d(n^{-1}\log n))$, where $d(x)=F^{-1}(x)$, $F(x)=\exp\{-\frac{1}{2}\pi\int_x^a[1/\varphi(t)]dt\}$. In the particular case that $cx^m<\varphi(x)<cx^m$, $m>1$ (contact of algebraic order), then $f^{(k)}(z)\varepsilon L(D;\alpha)$ implies that $\rho_n<c(\log n)^{-(k+\alpha)/(m-1)}$; and this result is sharp. *I. M. Sheffer* (State College, Pa.).

Mergelyan, S. N. On best approximation on closed sets. Doklady Akad. Nauk SSSR (N.S.) **62**, 163–166 (1948); errata, **63**, 220 (1948). (Russian)

In the paper reviewed above, estimates for ρ_n were established for regions with smooth boundary or boundary having a corner point or a "re-entrant" point. Now more general sets are considered. Part 1 is concerned with a finite region D whose boundary is also the complete boundary of the complement of $\bar{D}$. Let $d(t;r)$ be the distance from boundary point t to the image of the circle $|w|=r>1$ under the conformal mapping of $|w|>1$ on the complement of $\bar{D}$. (I) Let $\liminf_{n\to\infty}(\log\rho_n)/\log d(t;1+n^{-1})=A$. If $A<\infty$, then for every $\epsilon>0$, $f(z)\varepsilon L(D;A-\epsilon)$ [see the preceding review for definition of class L]; but if $A=\infty$, then $f(z)$ has an unbounded differential in $\bar{B}_t$ [$\bar{B}_t$ is defined like $\bar{B}_t$ in the second preceding review, with $z=0$ now replaced by t $(t\varepsilon\bar{D}-D)$]. If $A=0$, $f(z)$ may be nondifferentiable and not satisfy any Lipschitz condition of positive order, yet its modulus of continuity $\omega(\delta)$ satisfies in $\bar{B}_t$ the relation (1) $\omega(\delta)<C\cdot\min_{n\geq1}\{\rho_n+\delta/d(t;1+1/n)\}$; and (1) cannot be improved. (II) Let $A=\infty$, and set $M_n=\max|f^{(n)}(z)|$ $(z\varepsilon\bar{B}_t)$. Then for arbitrary positive integers $n_1<n_2<\cdots$, and arbitrary μ $(0<\mu<1)$,

$$(2)\quad M_n<C^n\cdot n!\sum_{k=1}^{\infty}\rho_{n_k}^{\mu}\{d(t;1+[(1-\mu)/n_{k+1}]\log(1/\rho_{n_{k+1}}))\}^{-n},$$

where C depends on d and on $\{\rho_n\}$. In particular, if $\rho_n<q^n$ $(q<1)$, then $M_n<C^n\cdot n!$, and $f(z)$ is analytic at t. (III) Let $\varphi(n)>0$ be any function for which $\lim_{n\to\infty}k^n\varphi(n)=\infty$ for every $k>1$, and let D be a finite region identical with the set of interior points of its closure. There exists a function $f(z)$, regular in D, continuous in $\bar{D}$, such that $\rho_n(D;f)<\varphi(n)$, but the boundary of D is a cut for $f(z)$.

Part 2 deals with the behavior of $\rho_n=\rho_n(E;f)$, where E is a closed nowhere dense set not separating the plane, and $f(z)$ is an arbitrary continuous function on E. (IV) Let $M(r)>0$ be any function increasing to ∞ for $r\to\infty$ fast enough so that $M(r)/r^N\to\infty$ for every N, and let E be an infinite set. There exists a positive function $\varphi(n)$, such that (3) $\rho_n(E;f)<\varphi(n)$ implies that $f(z)$ can be continued from set E to the whole plane, the function obtained by the resulting continuation being an entire function with $\max|F(z)|<M(r)$ $(|z|=r)$.

(V) Whatever be $\varphi(n)>0$, $n=1,2,\cdots$, and $\omega(\delta)>0$ $(\omega(\delta)\to0$ as $\delta\to0)$, there exists a perfect set of points $P\varepsilon[0,1]$ and a function $f(x)$ continuous on P, such that $\rho_n(P;f)<\varphi(n)$, $n>n_0$, yet for arbitrary a, b $(0\leqq a<b\leqq1)$, $\omega_{a,b}(\delta)>\omega(\delta)$, $\delta<\delta(a,b)$. Here $\omega_{a,b}(\delta)$ is the modulus of continuity of $f(x)$ on that portion of P contained in the segment $[a,b]$ $(\omega_{a,b}(\delta)=1$ if $P[a,b]=0)$. A classical result is extended as follows. (VI) Let E be a linear perfect set on $[0,1]$. If x_0 is a regular point (in the sense of the Dirichlet problem) of E, and if $\rho_n(E;f)<q^n$, $0<q<1$, then $f(x)$ is analytic at x_0. Finally, a result of S. Bernstein [C. R. Acad. Sci. Paris **177**, 937–939 (1923)] is extended. (VII) Let $E=[0,1]$. If $\rho_n(E;f)<q^n$ $(0<q<1)$ for an infinite sequence $n=n_1,n_2,\cdots$, and if $f(x)=0$ on a subset of E of positive capacity, then $f(x)=0$. *I. M. Sheffer* (State College, Pa.).

Zahorski, Z. On zeros of quasi-analytic (B) functions. Bull. Calcutta Math. Soc. **39**, 157–165 (1947).

If Z is a set of complex numbers, and $\{\epsilon_k\}$ a set of positive quantities with $\lim\epsilon_n=0$, the author says that $f(z)$, defined on Z, is quasi-analytic $(B\epsilon_n)$ on Z, if there exist an increasing sequence of positive integers $\{n_k\}$ and a sequence of polynomials $\{P_k(z)\}$, P_k being of degree n_k, such that on Z: $|f(z)-P_k(z)|<\epsilon_{n_k}$ $(k\geqq1)$. The author generalizes a result of Mazurkiewicz and Szmuszkowiczowna in proving the following. To each $\{\epsilon_n\}$ and each perfect set E of points z with $|z|=1$, there corresponds a function $f(z)$ quasi-analytic $(B\epsilon_n)$ on $|z|\leqq1$, possessing all derivatives quasi-analytic $(B\epsilon_n)$, not identically zero, such that $f(z)=f^{(n)}(z)=0$ for every n and every $z\varepsilon E_1\subset E$ where E_1 is of the power of the continuum. *S. Mandelbrojt* (Paris).

***Kjellberg, Bo. On certain integral and harmonic functions. A study in minimum modulus.** Thesis, University of Uppsala, 1948. 64 pp.

Let a and b denote real numbers satisfying $1<a<b$. The author is concerned with the class of functions u which are continuous in the finite plane, vanish on all the closed intervals $[b^n,ab^n]$ (n an integer) of the real axis and are harmonic and positive elsewhere. It is shown that there is just one normalized member $(u(-1)=1)$ in the class. The functions of the class satisfy a simple functional equation. The dependence of the order $\rho(a,b)$ of u on the parameters a and b is studied and the following results are obtained: (1) if a and b tend to infinity so that $(\log a)/(2\log b)=\sigma$ $(0<\sigma<\frac{1}{2})$, then $\lim\rho(a,b)=\sigma$; (2) if a and b decrease to one on the same locus, then $\lim\rho(a,b)=\frac{1}{2}$ independently of σ $(0<\sigma<\frac{1}{2})$. Use is made of simple auxiliary harmonic functions to establish these results. With each admitted u there is associated an entire function f of order $\rho(a,b)$, mean type, and of very regular growth. In this manner the author establishes the existence of admitted f for which the logarithmic density of the set of r for which $\min_{|z|=r}|f(z)|>$constant exists and is arbitrarily close to $1-2\rho$. This result establishes the sharpness of a certain theorem concerning the minimum modulus of an entire function φ ($\not\equiv$constant) of order ρ $(<\frac{1}{2})$. This theorem, which is a direct consequence of an inequality due to Beurling [thesis, Uppsala, 1933], states that the lower logarithmic density of the set of r for which $\min_{|z|=r}|\varphi(z)|>$constant is not less than $1-2\rho$.

Problems concerning entire functions bounded on a given set are treated. The following theorem is shown with the

aid of Beurling's inequality cited above. Let f be entire with positive zeros (if any) and (1) $|f(z)| \leqq 1$ for $z \in B$, B closed and on $(\Im z = 0, \Re z \geqq 1)$, (2) $\log M(r) \leqq A(r)$, where $A(r)$ is monotone increasing, (3) $\int^{\infty} \varphi^{-1} dA < +\infty$, where φ is a positive continuous increasing function and $\varphi(r) = o(r)$. Let B_φ denote the set of $x \geqq 1$ for which an $r \in B$ exists such that $|x - r| \leqq r/\varphi(r)$, and $m_l B_\varphi(1, r)$ the logarithmic measure of the part of B_φ in the interval $[1, r]$. Then $\limsup_{r \to \infty} \{\frac{1}{2} m_l B_\varphi(1, r) - \log A(r)\} = +\infty$ implies $f(z) \equiv$ constant.

A final section is concerned with special classes of minimal positive harmonic functions. *M. Heins.*

Collingwood, E. F. Sur certains ensembles définis pour les fonctions méromorphes. C. R. Acad. Sci. Paris **227**, 615–617 (1948).

Première note d'une série dans laquelle l'auteur étudie la relation entre les valeurs déficientes d'une fonction $f(z)$ méromorphe pour $|z| < R \leqq \infty$ et le comportement de la fonction dans certains domaines. Par $G(a, \sigma)$ on désigne l'un quelconque des domaines dans lesquels $|f(z) - a| < \sigma$ si a est fini et $|f(z)| \sigma > 1$ si a est infini; $G(a, \sigma)$ est dit borné si sur toute sa frontière $|z| < R$. Par $\lambda(r)$ on désigne le nombre des $G(a, \sigma)$ qui coupent $|z| = r < R$; $G_\nu(r, a, \sigma)$ désigne l'un de ces domaines, $\nu \leqq \lambda(r)$; $p_\nu(r, a, \sigma)$ est la valence, finie ou infinie, de $f(z)$ dans $G_\nu(r, a, \sigma)$. L'auteur étend ces définitions au cas où σ est remplacé par une fonction $\sigma(r)$ qu'il appelle niveau pour a et $f(z)$; la valence $P(r, a, \sigma(r))$ est le maximum de $p_\nu(r, a, \sigma(r))$ pour $\nu \leqq \lambda(r)$. Le niveau peut être nul, ou fermé, ou ouvert. On désigne par $V(a, \sigma(r), p(r))$, où $p(r)$ est donné, l'ensemble des r pour lesquels $P(r, a, \sigma(r)) \leqq p(r)$ et par $E(a, \sigma(r), p(r))$, le sous ensemble de V pour lequel $\sigma(r)$ est un niveau nul ou fermé. Par $\mathfrak{V}(a, \sigma(r))$ et $\mathfrak{d}(a, \sigma(r))$ on désigne les sommes des $V(a, \sigma(r), q)$ et $E(a, \sigma(r), q)$ pour q entier. Les ensembles complémentaires $CV(a, \cdots), \cdots$, sont aussi introduits. Dans le cas parabolique, $R = \infty$, un théorème d'Iversen [Ofversigt af Finska Vetenskaps-Societetens Förhandlingar **64**, no. 4 (1921), p. 5] entraîne

$$V(a, \sigma(r), p(r)) \subseteq E(a, \sigma(r)/k, p(r)), \quad k > 1.$$

G. Valiron (Paris).

Collingwood, E. F. Une inégalité dans la théorie des fonctions méromorphes. C. R. Acad. Sci. Paris **227**, 709–711 (1948).

Utilisant les notions et notations introduites dans l'analyse précédente, l'auteur généralise sous la forme suivante un de ses théorèmes antérieurs [mêmes C. R. **179**, 1125–1127 (1924)] complété ultérieurement par Teichmüller [Deutsche Math. **2**, 96–107 (1937)] et Selberg [Comment. Math. Helv. **18**, 309–326 (1946); ces Rev. **8**, 23]. Soit $f(z)$ méromorphe pour $|z| < R \leqq \infty$. S'il existe a et $\sigma(r)$ ($\sigma(r)$ fixe ou monotone tendant vers zéro) tels que R soit point limite de $E = E(a, \sigma(r), p(r))$, on a

$$(1) \qquad m(r, a) < K(r) p(r) + \log^+ \{1/\sigma(r)\} + O(1),$$

$K(r)$ étant égal à $\pi + \log r$ si $R = \infty$ et à $\pi + o(1)$ si R est fini. Si, en outre, il existe des nombres A, B, $0 < A < B < R$, tels que, pour $A < r < B$ aucun $G_\nu(r, a, \sigma(r))$, $\nu \leqq \lambda(r)$, ne contienne une courbe fermée entourant la circonférence $|z| = B$, l'inégalité (1) vaut pour r appartenant à E, avec $K(r) = \pi/2$. La fonction $m(r, a)$ est la moyenne de Nevanlinna.

G. Valiron (Paris).

Collingwood, E. F. Inégalités relatives à la distribution des valeurs d'une fonction méromorphe dans le plan fini. C. R. Acad. Sci. Paris **227**, 749–751 (1948).

Dans cette troisième note [cf. l'analyse précédente], concernant le cas parabolique, $R = \infty$, l'auteur introduit les notions d'indice. Si $T(r, f)$ est la fonction caractéristique de Nevanlinna, il considère les limites d'indétermination pour r infini de $\log^+ \{1/\sigma(r)\}/T(r, f)$. La limite supérieure $\bar{I}(\sigma(r), f)$ est l'indice supérieure, la limite inférieure $\underline{I}(\sigma(r), f)$ est l'indice inférieure, et, si ces limites sont égales, leur valeur $I(\sigma(r), f)$ est l'indice de $f(z)$ et du niveau $\sigma(r)$. La limite inférieure $\delta(a)$ de $m(r, a)/T(r, f)$ est la déficience de Nevanlinna, la limite supérieure $\Delta(a)$ est la déficience supérieure (de Valiron). L'auteur donne deux théorèmes. (I) On a $\Delta(a) \leqq \bar{P} + \bar{I}(\sigma(r), f)$, $\delta(a) \leqq \bar{P} + \underline{I}(\sigma(r), f)$, $\delta(a) \leqq \underline{P} + \bar{I}(\sigma(r), f)$, où $\bar{P}$ et $\underline{P}$ sont respectivement les limites supérieure et inférieure de $\{P(r, a, \sigma(r)) \log r\}/T(r, f)$. (II) Si, en outre, $f(z)$ si a est infini et $1/(f(z) - a)$ si a est fini, est borné sur une courbe allant à l'infini, on peut remplacer dans les inégalités précédentes $\bar{P}$ et $\underline{P}$ respectivement par les limites supérieure et inférieure de $\frac{1}{2}\pi P(r, a, \sigma(r))/T(r, f)$. L'auteur en déduit dans certains cas des inégalités très précises pour la valence. *G. Valiron* (Paris).

Fuks, B. A. Cauchy's integral formula in the theory of analytic functions of several complex variables. Uspehi Matem. Nauk (N.S.) **2**, no. 3(19), 142–157 (1947). (Russian)

The author gives a review of generalizations (by Bergman, Bochner, A. Weil and others) of Cauchy's integral formula to the case of several complex variables and of some related questions. In the opinion of the reviewer the article does not give a true picture of the topic. The Cauchy formula is an integral expression reproducing analytic functions in the case of one variable; its kernel is independent of the domain; this is no longer the case for several variables. Since, in addition, the integration has to be carried out over special surfaces or hypersurfaces, these formulae lose considerably in value as means of investigation, when we pass to several variables. On the other hand there exist other methods in several variables which lead to the desired results. This situation is not clarified in the review, which has only a compilatory and formal character. The author does not even discuss the relation of the generalized Cauchy formulae to other residue formulae, not to mention their relations to the generalized Green's formulae, etc. Some essential bibliographical data are missing, e.g., the author does not indicate that formulae expressing the function in terms of its value on the boundary hypersurface and on the distinguished boundary surface were given by Bergman [Revista Ci. Lima **43**, 675–682 (1941); **44**, 131–140, 377–394 (1942), in particular, p. 389; these Rev. **4**, 140; Jber. Deutsch. Math. Verein **42**, 238–252 (1933), respectively]; he cites only later papers. *K. Zarankiewicz.*

Beckenbach, E. F. A Looman-Menchoff theorem for Newtonian vectors. Univ. Nac. Tucumán. Revista A. **6**, 319–329 (1948).

A vector function $\mathbf{X}(x, y, z) = \mathbf{i}X_1 + \mathbf{j}X_2 + \mathbf{k}X_3$ having continuous partial derivatives of the first order in a finite three-dimensional domain D has been called [Fulton and Rainich, Amer. J. Math. **54**, 235–241 (1932)] a Newtonian vector function if and only if its curl $\nabla \times \mathbf{X}$ and its divergence $\nabla \cdot \mathbf{X}$ both vanish identically in D. Thus the class of Newtonian vectors may be considered as a generalization of the

class of analytic functions of a complex variable. In this paper the author shows that the defining properties can be weakened without broadening the class of functions; the hypotheses are weakened in the direction of the classic Looman-Menchoff theorem [Saks, Theory of the Integral, Warszawa-Lwów, 1937, p. 139], and in a recent result due to the author [Bull. Amer. Math. Soc. 51, 240–245 (1945); these Rev. 6, 227]. The principal obstacle to be overcome is that of replacing differentiability requirements; this is done by using hypotheses of total differentiability. The principal result is the following one. If the vector function $\mathbf{X}(x, y, z)$ is continuous in a finite domain D, if, except at most at the points of a denumerable set of points in D, the components of $\mathbf{X}$ are totally differentiable in the planes parallel to the coordinate planes, and if $\nabla\times\mathbf{X}=0$ and $\nabla\cdot\mathbf{X}=0$ almost everywhere in D, then $\mathbf{X}(x, y, z)$ is a Newtonian vector function in D. *M. Reade.*

Degtereva, M. Elements of a theory of analytic functions on the kernel of linear algebras. Doklady Akad. Nauk SSSR (N.S.) **60**, 1491–1493 (1948). (Russian)

In an associative and distributive algebra of order n over the field of real numbers, with a commutative subalgebra θ of order $k\leqq n$, there is presented a theory of analytic functions with the independent variable restricted to range over the kernel θ, generalizing the notion of an analytic function of a complex variable beyond the known generalization to commutative algebras. At a given point z of θ, right and left hand derivatives of a function $f(z)$ are given by

$$\overrightarrow{f'(z)}=\left(\sum_{s=1}^{n}\frac{\partial f_s}{\partial z_j}e_s\right)\frac{1}{e_j},\qquad \overleftarrow{f'(z)}=\frac{1}{e_j}\sum_{s=1}^{n}\frac{\partial f_s}{\partial z_j}e_s,$$

$j=1, 2, \cdots, k$. In a commutative algebra, these derivatives are equal. The author obtains generalizations of the Cauchy-Riemann equations as necessary and sufficient conditions for right and left hand derivatives, and establishes analogues of other function-theoretic results including the Cauchy integral theorem. *E. F. Beckenbach.*

Degtereva, M. On a question in the construction of a theory of analytic functions in linear algebras. Doklady Akad. Nauk SSSR (N.S.) **61**, 13–15 (1948). (Russian)

The effort to construct in a noncommutative linear algebra an analogue of the class of analytic functions of a complex variable, without restricting the range of the independent variable to the kernel θ of the algebra, leads to the necessity of starting with a more general definition of analyticity than is customary in the classical theory. One such definition is that of Hausdorff. A function $f(z)$ of the hypercomplex variable z in a linear algebra is said to be analytic in the sense of Hausdorff provided the differential

$$df=\sum_{s=1}^{n}df_se_s=\sum_{s=1}^{n}\left(\sum_{j=1}^{n}\frac{\partial f_s}{\partial z_j}dz_j\right)e_s,$$

the existence of which is assumed, is a linear homogeneous function of the differential dz and therefore has the form $df=\sum_{s=1}^{n}u_s\,dz\,v_s$, where u_s and v_s are functions of the hypercomplex variable z. In the present paper it is shown that in any commutative algebra any function analytic in the sense of Hausdorff at a point is differentiable in the Cauchy-Riemann sense [see the preceding review] at the same point. Examples of generalizations of the Cauchy-Riemann equations in the direction of the Hausdorff definition are given for certain noncommutative algebras.

E. F. Beckenbach (Los Angeles, Calif.).

Theory of Series

Frink, Orrin. A ratio test. Bull. Amer. Math. Soc. **54**, 953 (1948).

The author proves the following theorem and a natural generalization. A series of positive terms $\sum a_n$ converges if $\lim_{n\to\infty}(a_n/a_{n-1})^n<1/e$ and diverges if $\lim_{n\to\infty}(a_n/a_{n-1})^n>1/e$. The theorems are equivalent to "Raabe's test," from which they readily follow. *T. Fort* (Athens, Ga.).

Agnew, Ralph Palmer. Methods of summability which evaluate sequences of zeros and ones summable C_1. Amer. J. Math. **70**, 75–81 (1948).

The author obtains sufficient conditions for a Toeplitz method (or a method of a somewhat more general class) to be regular, i.e., to evaluate every convergent sequence to its limit. The conditions are expressed entirely in terms of sequences of 0's and 1's that are evaluated by the Cesàro method C: $\sigma_n=(s_1+\cdots+s_n)/n$. The method A is shown to be regular whenever it satisfies one of the following three conditions. (1) If s is a divergent sequence of 0's and 1's evaluated by C, then A evaluates s to the same limit; (2) A evaluates s to $\frac{1}{2}$ whenever s is a sequence of 0's and 1's ending in 0, 1, 0, 1, 0, 1, $\cdots$; and to zero whenever s is a divergent sequence of 0's and 1's evaluated to zero by C; (3) A evaluates s to $\frac{1}{2}$ whenever s is a sequence of 0's and 1's evaluated to $\frac{1}{2}$ by C. *G. Piranian.*

Mears, Florence M. Transformations of double sequences. Amer. J. Math. **70**, 804–832 (1948).

H. J. Hamilton [Duke Math. J. 2, 29–60 (1936)] derived conditions on the $2n$-dimensional matrix of complex numbers $\|a_{m_1\cdots m_n k_1\cdots k_n}\|$ necessary and sufficient for the transforming of all n-dimensional sequences $\{s_{k_1\cdots k_n}\}$ of class U into n-dimensional sequences $\{t_{m_1\cdots m_n}\}$ of class V, by means of the relation

$$t_{m_1\cdots m_n}=\sum_{k_1,\cdots,k_n=1}^{\infty}a_{m_1\cdots m_n k_1\cdots k_n}s_{k_1\cdots k_n},$$

for each U and V of a set of 16 classes of sequences of complex numbers. The latter range from those which are regularly convergent with all row limits and the principal limit equal to zero, to those which are merely ultimately bounded. In the present paper the author considers the case $n=2$ and introduces four more classes of double sequences, among which, for example, is the class of rectangular partial sums of absolutely convergent double series. Theorems corresponding to those of Hamilton are obtained for the transformation of a specific one of the last four types into any one of the twenty possible types. *J. D. Hill.*

Minakshisundaram, S., and Rajagopal, C. T. An extension of a Tauberian theorem of L. J. Mordell. Proc. London Math. Soc. (2) **50**, 242–255 (1948).

Let $A^r(\omega)=\sum_{\lambda_n\leqq\omega}(\omega-\lambda_n)^r a_n$, $r>0$, be the Riesz (λ_n, r)-sum of a series $\sum a_n$. The authors prove the following generalization of a Tauberian theorem of Mordell [J. London Math. Soc. 3, 86–89 (1928)] involving Cesàro means of the first order. If $\phi(x)$, $\psi(x)$ are positive functions of $x>0$ such that $\theta(x)=\{\phi(x)/\psi(x)\}^{1/(r+1)}\leqq x$, regularly oscillating for $x\to\infty$ in a certain sense, then the conditions (i) $\lambda_n-\lambda_{n-1}\leqq C_1\theta(\lambda_n)$, (ii) $a_n\leqq C_2(\lambda_n-\lambda_{n-1})\psi(\lambda_n)$, (iii) $|A^r(\omega)|\leqq C_3\phi(\omega)$ imply $A^1(\omega)=O\{\phi(x)\}^{1/(r+1)}\{\psi(x)\}^{r/(r+1)}$ for $\omega\to\infty$. There are applications and variants. For instance, if $a_n\leqq C(\lambda_n-\lambda_{n-1})\lambda_n^{\alpha}$, $A^r(\omega)=o(\omega^{\beta})$, $r>0$, $0\leqq\beta\leqq\alpha+r+1$,

then $A^1(\omega)=o(\omega^{(\alpha r+\beta)/(r+1)})$ provided

(iv) $$\lambda_n-\lambda_{n-1}=o(\lambda_n^{(\beta-\alpha)/(r+1)}).$$

This is a one-sided analogue, with the additional condition (iv), to a two-sided theorem of K. Ananda Rau [same Proc. (2) **34**, 414–440 (1932)]. *G. G. Lorentz* (Tübingen).

Rajagopal, C. T. A series associated with Dirichlet's series. Acta Univ. Szeged. Sect. Sci. Math. **11**, 201–206 (1948).

Let

$$0=D_0<D_1<\cdots<D_n\to\infty$$

and assume that $d_n=D_n-D_{n-1}$ is $O(1)$. Set $\lambda_n=\log D_n$ and $c_n=d_n/D_n$. The author discusses the association of a factorial series (1) $\sum_1^\infty a_n\prod_1^n(1+c_ks)^{-1}$ with the Dirichlet series (2) $\sum_1^\infty a_n\exp(-\lambda_ns)$. It is first shown that (1) and (2) are equiconvergent, both pointwise and uniformly, excluding the points $s=-1/c_n$. Turning to Abel summability, a standard Tauberian theorem for (2) is applied to show that if $F(s)$ is the sum of (1) and if $F(s)\to C$ as $s\downarrow 0$, then if $a_n=O(c_n/\lambda_n)$, $\sum_1^\infty a_n=C$. Similarly, if $a_n\geqq 0$ and $F(s)\sim Cs^{-\alpha}\prod_1^k(\ln_k s^{-1})^{\alpha_k}$ as $s\downarrow 0$, where $\ln_k$ is the kth iterate of the logarithm function, then

$$\sum_1^n a_k\sim C\prod_1^m(\ln_k(D_n)^{\alpha_k-1}/\Gamma(\alpha_0+1).$$

For the special case where $F(s)\sim Cs^{-\alpha}$ as $s\downarrow 0$ and $0<\alpha\leqq 1$, the condition $a_n\geqq -Kc_n/(\lambda_n)^{1-\alpha}$ is sufficient to assure $\sum_1^n a_k\sim C(\lambda_n)^\alpha/\Gamma(\alpha+1)$. It is remarked that corresponding results hold for the associated series $\sum_1^\infty a_n\prod_1^n(1-c_ks)$.

R. C. Buck (Providence, R. I.).

Hsu, L. C. A theorem on the asymptotic behavior of a multiple integral. Duke Math. J. **15**, 623–632 (1948).

The author obtains an asymptotic formula for multiple integrals of the form

$$\int\cdots\int_D\varphi(u_1,\cdots,u_n)[f(u_1,\cdots,u_n)]^N du_1\cdots du_n,$$

as $N\to\infty$. This generalizes the 1-dimensional case due to Laplace [cf. Pólya and Szegö, Aufgaben und Lehrsätze aus der Analysis, v. 1, Springer, Berlin, 1925, p. 78, no. 201]. The main theorem deals with the case where the maximum value of f is assumed on the boundary, rather than the interior, of the domain D. *P. Hartman* (Baltimore, Md.).

Chatterjee, Anunoy. On a continued fraction of a general type. Bull. Calcutta Math. Soc. **40**, 69–75 (1948).

This paper generalizes the matrix product formulas of Upadhyay and Das Gupta [same Bull. **39**, 65–70 (1947); these Rev. **9**, 508] and Milne-Thomson [The Calculus of Finite Differences, Macmillan, London, 1933, pp. 109–110]. Let

$$a\begin{pmatrix}1&2&3&\cdots&n\\r&s&t&\cdots&v\end{pmatrix}=a_1(r)+\cfrac{1}{a_2(s)+\cfrac{1}{a_3(t)+\cdots+\cfrac{1}{a_n(v)}}},$$

where

$$a_i(m)=a_{i1}+\cfrac{1}{a_{i2}+\cfrac{1}{a_{i3}+\cdots+\cfrac{1}{a_{im}}}},$$

and let p and q denote the numerators and denominators, respectively, of the convergents. Let the operator

$$\overrightarrow{\begin{bmatrix}p\\q\end{bmatrix}}=\begin{bmatrix}p&q\\q&0\end{bmatrix}.$$

Then

$$\begin{bmatrix}p\begin{pmatrix}1&2&\cdots&n\\r&s&\cdots&v\end{pmatrix}\\ q\begin{pmatrix}1&2&\cdots&n\\r&s&\cdots&v\end{pmatrix}\end{bmatrix}=\overrightarrow{\begin{bmatrix}p_1(r)\\q_1(r)\end{bmatrix}}\overrightarrow{\begin{bmatrix}p_2(s)\\q_2(s)\end{bmatrix}}\cdots\overrightarrow{\begin{bmatrix}p_{n-1}(u)\\q_{n-1}(u)\end{bmatrix}}\begin{bmatrix}p_n(v)\\q_n(v)\end{bmatrix}$$

$$=\overrightarrow{\prod_{j=1}^{r-1}\begin{bmatrix}a_{1j}\\1\end{bmatrix}}\begin{bmatrix}a_{1r}\\1\end{bmatrix}\overrightarrow{\prod_{j=1}^{s-1}\begin{bmatrix}a_{2j}\\1\end{bmatrix}}\begin{bmatrix}a_{2s}\\1\end{bmatrix}\cdots$$

$$\times\overrightarrow{\prod_{j=1}^{u-1}\begin{bmatrix}a_{(n-1)j}\\1\end{bmatrix}}\begin{bmatrix}a_{(n-1)u}\\1\end{bmatrix}\overrightarrow{\prod_{j=1}^{v-1}\begin{bmatrix}a_{nj}\\1\end{bmatrix}}\begin{bmatrix}a_{nv}\\1\end{bmatrix}.$$

Methods are indicated for the extension of these formulas to continued fractions whose elements are continued fractions with elements which are themselves continued fractions.

E. Frank (Chicago, Ill.).

Fourier Series and Generalizations, Integral Transforms

Alaci, V. Une classe de fonctions simplement-discontinues. Bull. Sci. Tech. Polytech. Timişoara **13**, 1–17 (1948).

The author shows that $I^n(A)=\int_0^\pi(u^{-1}\sin Au\cos^n u)du$, $n=0,1,\cdots$, takes $2n+3$ values for $-\infty<A<\infty$ and gives examples of functions with simple jumps which can be written as $I_n(a(x,y))=b(x,y)$. *P. Civin* (Eugene, Ore.).

Pagni, M. Un'osservazione sui coefficienti di Fourier delle funzioni crescenti. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **4**, 672–675 (1948).

In connection with a well-known result due to C. Carathéodory, necessary and sufficient conditions are given in order that a sequence of numbers $a_1, b_1, a_2, b_2, \cdots$, be the Fourier coefficients of a strictly increasing function in $(0, 2\pi)$.

O. Szász (Cincinnati, Ohio).

Ditkin, V. A. On a question about the formal multiplication of trigonometric series. Doklady Akad. Nauk SSSR (N.S.) **60**, 1495–1498 (1948). (Russian)

For $f(x)=\sum_{-\infty}^\infty c_ne^{-inx}$ and $\alpha_{m+n}\leqq Q\alpha_m\alpha_n$, $1\leqq\alpha_n\leqq Cn^r$, let $\sum_{-\infty}^\infty c_n\alpha_n<\infty$ and $|s_n|\leqq\alpha_n$. Put

$$\sum_{-\infty}^{\infty}c_ks_{k+r}=\sigma_n,\quad s_0\xi^{r+2}/(r+2)!+\sum_{-\infty}^{\infty}s_ne^{-in\xi}/(-in)^{r+2}=s(\xi),$$

$$\sigma_0\xi^{r+2}/(r+2)!+\sum_{-\infty}^{\infty}\sigma_ne^{-in\xi}/(-in)^{r+2}=\sigma(\xi).$$

Using Gelfand's [Rec. Math. [Mat. Sbornik] N.S. **9**(51), 51–66 (1941); these Rev. **3**, 51] theory of normed rings of trigonometric series, the author proves the following theorem. If $\sigma(\xi)$ is in (a, b) a polynomial of order not exceeding $r+1$, then $s(\xi)$ is a polynomial of order not exceeding $r+1$ in every interval complementary to $(a, b)\cap F$, where F denotes the set of zeros of $f(x)$. A corollary of this is the following. In order that the formal product of $\sum c_ne^{-inx}$ and $\sum s_ne^{-inx}$ converge everywhere to zero, it is necessary that $s(\xi)$ be a polynomial of order not exceeding $r+1$ in every interval complementary to the set of zeros of $f(x)$.

František Wolf (Berkeley, Calif.).

Salem, R., and Zygmund, A. A convexity theorem. Proc. Nat. Acad. Sci. U. S. A. **34**, 443–447 (1948).

Littlewood and Paley proved [cf. Proc. London Math. Soc. (2) **43**, 105–126 (1937)] that, if $S_n(x)$ is the partial sum of order n of the Fourier series of the function $g(x)\in L^p$ $(1<p\leqq 2)$ and if $n(x)$ is any measurable function taking nonnegative integral values, then

$$(1)\qquad \int_0^{2\pi}\frac{|S_{n(x)}(x)|^p}{\log[n(x)+2]}dx\leqq A_p\int_0^{2\pi}|g(x)|^p dx.$$

Their proof is long and difficult. For $p=2$, there exists a simple proof, due to Hardy and Littlewood. A simpler proof of the general case, based on some theorem analogous to M. Riesz's convexity theorem, seems to be impossible at first sight, because (1) does not hold for $p=1$.

However, if one considers power series $f(z)=\sum_{k=0}^{\infty}c_kz^k$ of Hardy's class H^p, then the analogous inequality (with $g(x)=f(e^{ix})$ and with the partial sums of the series $\sum_{k=0}^{\infty}c_ke^{ikx}$), while equivalent to (1) for $p=2$, holds also for $p=1$ [cf. Zygmund, Fund. Math. **30**, 170–196 (1938)]. The two extremal cases $p=1, 2$ being relatively simple, one can pass to an arbitrary p $(1\leqq p\leqq 2)$ by a convexity argument, derived from the Riesz-Thorin convexity theorem [cf. Thorin, Comm. Sem. Math. Univ. Lund [Medd. Lunds Univ. Mat. Sem.] **9**, 1–58 (1948); these Rev. **10**, 21]. One can return to (1), in the case $1<p\leqq 2$, since it is known that the property H^p for a power series is equivalent to the property L^p for the Fourier series of its real part, provided that $p>1$.

The convexity property in question reads as follows. Let $f(z)=\sum_{k=0}^{\infty}x_kz^k$ be regular for $|z|<1$ and belong to the class $H^{1/\alpha}$ $(\alpha>0)$. Let $M(\alpha,\beta)$ denote the maximum of the finite complex bilinear form $\sum_{j=0}^{m}\sum_{k=0}^{n}a_{jk}x_jy_k$ when the complex variables x_j, y_k are subject to the conditions $\sum_{k=0}^{n}|y_k|^{1/\beta}\leqq 1$ $(\beta\geqq 0$; in the case $\beta=0$ this means that $\max|y_k|\leqq 1)$ and $\int_0^{2\pi}|f(re^{i\theta})|^{1/\alpha}d\theta\leqq 1$ $(\alpha>0)$, the second condition holding for every $f(z)$ belonging to $H^{1/\alpha}$ and having $x_0,\cdots,x_m$ as its $m+1$ first expansion coefficients, and for every nonnegative $r<1$. Then

$$M(t\alpha_1+(1-t)\alpha_2,\ t\beta_1+(1-t)\beta_2)\leqq C(\alpha_1,\alpha_2)M^t(\alpha_1,\beta_1)M^{1-t}(\alpha_2,\beta_2)$$

for $0\leqq t\leqq 1$. *B. de Sz. Nagy* (Szeged).

Nikol'skiĭ, S. M. On linear methods of summation of Fourier series. Izvestiya Akad. Nauk SSSR. Ser. Mat. **12**, 259–278 (1948). (Russian)

The paper is concerned with a general theory of summation of trigonometric series. If

$$U_n(f,x)=\tfrac{1}{2}a_0+\sum_{k=1}^{n}\lambda_k^{(n)}(a_k\cos kx+b_k\sin kx),$$

then a necessary and sufficient condition that

$$\lim_n U_n(f,x)=f(x)$$

for any continuous f (for any Lebesgue point of an integrable f) is (A) $\lim_n\lambda_k^{(n)}=1$ and

$$(B)\qquad \int_0^{\pi}\Big|\tfrac{1}{2}+\sum_{k=1}^{n}\lambda_k^{(n)}\cos kt\Big|dt\leqq M.$$

This follows from classical theorems of Banach on weak convergence in B-spaces. The paper is devoted to finding conditions which $\lambda_k^{(n)}$ have to satisfy in order that (B) be true. It is shown that (B) implies the existence of a C such that (α) $|\lambda_k^{(n)}|\leqq C$ and (β) $|\sum_{k+1}^{n}\lambda_{n-1-k}^{n}/k|\leqq C$. An example shows that (α), (β) are not equivalent to (B). But if, for every n, $\lambda_k^{(n)}$ is convex (concave), this equivalence holds. *František Wolf* (Berkeley, Calif.).

Hsiang, Fu-Cheng. The Cesàro summability of negative order for Fourier series. Acad. Sinica Science Record **2**, 164–171 (1948).

W. H. Young's criterion for the convergence of a Fourier series may be expressed in the form: if (i) $\int_0^t d\{u\varphi(u)\}=O(t)$ in $(0,\pi)$, and (ii) the Fourier series of the even function $\varphi(t)$ at $t=0$ is summable (C), or (A), then the series is summable $(C,-1+\delta)$, for every $\delta>0$ [cf. Hardy and Littlewood, Proc. London Math. Soc. (2) **28**, 301–311 (1928). The (C) and (A) hypotheses are equivalent in virtue of the boundedness of $\varphi(t)$]. L. C. Young proved that if W. H. Young's hypothesis (i) is replaced by the hypothesis that $\varphi(t)$ is of bounded pth power variation in $(0,\pi)$, $p>1$, and condition (ii) is dropped, then the conclusion holds with summability $(C,-p^{-1}+\delta)$, $\delta>0$.

A. C. Offord and the reviewer gave a localized form of W. H. Young's theorem with (i) assumed only for an interval $(0,\eta)$, $\eta<\pi$, with the Fourier coefficients assumed to be $o(n^{-\gamma})$, $0<\gamma<1$, and with summability $(C,-\gamma)$ in the conclusion [Bosanquet and Offord, Proc. London Math. Soc. (2) **40**, 273–280 (1935); Bosanquet, Quart. J. Math., Oxford Ser. **6**, 113–123 (1935)]. Here the author replaces L. C. Young's hypothesis by a local one, with the coefficients assumed to be $O(n^{-\alpha})$, $0<\alpha<p^{-1}$, with condition (ii) reimposed, and with summability $(C,-\alpha+\delta)$, $\delta>0$, in the conclusion. His proof contains a number of minor errors and some omissions, but is broadly correct.

It should be remarked here that the result of Offord and the reviewer is also a corollary of a localization theorem of A. Zygmund, which they had overlooked [Zygmund, Math. Z. **24**, 47–104 (1925), pp. 100–101; Trigonometrical Series, Warzawa-Lwów, 1935, chap. 8, ex. 12, p. 305]. By using Zygmund's theorem the author's result may be improved: if the coefficients are assumed to be $o(n^{-\gamma})$, $0<\gamma<p^{-1}$, and condition (ii) is dropped, the conclusion is summability $(C,-\gamma)$. *L. S. Bosanquet* (London).

Loo, Ching-Tsün. On the Cesàro summability of lacunary partial sums of Fourier series. Tôhoku Math. J. **48**, 215–220 (1941).

Cheng, Min-Teh. Note on Cesàro summability of Fourier series. Tôhoku Math. J. **48**, 282–291 (1941).

Let $k>1$ be an integer and let $S_n(x)$ be the partial sums of a periodic and L-integrable function $f(x)$. It is shown that almost everywhere (more precisely, for every x for which $\int_0^h\varphi(t)dt=o(h)$, $\int_0^h|\varphi(t)|dt=O(h)$, with $\varphi(t)=f(x+t)+f(x-t)-2f(x))$ the sequence $S_{n_k}(x)$, $n_k=n^k$, is summable (C,α), $\alpha>0$, to limit $f(x)$. For $k=3,4$ the result is proved by Cheng, and is extended to general integral k by Loo. The latter also proves that almost everywhere $\tilde{S}_{n_k}(x)\to\tilde{f}(x)(C,\alpha)$, $\alpha>0$, where $\tilde{S}_n(x)$ is the polynomial conjugate to $S_n(x)$ and $\tilde{f}(x)$ the function conjugate to $f(x)$. [This type of problem originated with Zalcwasser, Studia Math. **6**, 82–88 (1936).] *A. Zygmund* (Chicago, Ill.).

Misra, M. L. On the summability of the conjugate series of a Fourier series by logarithmic means. Proc. Nat. Inst. Sci. India **13**, 157–168 (1947).

The author obtains a scale of theorems connecting the logarithmic means of a function with the logarithmic means

of the conjugate series of its Fourier series. The simplest case is the following theorem. Let (1) $\int_0^t|\psi(u)|\,du=O(-t\log t)$ and (2) $g(t)=\pi^{-1}\int_t^\pi\psi(u)\cot\frac{1}{2}u\,du=o(-\log t)$ as $t\to+0$. Then a necessary and sufficient condition that the conjugate series of the odd function $\psi(t)$ at $t=0$ be summable by Riesz's logarithmic means of order 1, or summable $(R,1)$, to zero is that (3) $\int_t^\pi u^{-1}g(u)du=o(-\log t)$.

The author states that this can be deduced from a theorem of A. C. Offord and the reviewer [Compositio Math. 1, 180–187 (1935)], which arose out of a theorem of G. H. Hardy on Fourier series [Quart. J. Math., Oxford Ser. 2, 107–112 (1931)]. He also constructs an example of a function satisfying (2), and incidentally (1), for which the logarithmic means of the conjugate series diverge. An example serving the same purpose is $\psi(t)=(-\log t)^{-1}$, which satisfies (1) and (2), but not (3).

It should be mentioned that Offord and the reviewer stated that the $(R,1)$ summability is a consequence of (1) and (3) alone. Their argument, which was not given in detail, runs as follows: (1) implies that (4) $nb_n=O(\log n)$ $(C,1+\delta)$, $\delta>0$; (3) implies that (5) $\sum_{\nu=1}^n\beta_\nu/\nu=o(\log n)$ (C), where $\beta_n=b_1+\cdots+b_n$; (4) and (5) together imply that (6) $\sum_{\nu=1}^n\beta_\nu/\nu=o(\log n)$, which is equivalent to summability $(R,1)$ to zero. Thus (2) is superfluous in the sufficiency part. On the other hand to obtain (3) as a necessary condition we may argue: it follows from the summability $(R,1)$ that (7) $\int_t^\pi u^{-1}g(u)du=o(-\log t)$ (C), where the (C) is in fact (C,δ), $\delta>0$; (7) implies (3) if and only if (8) $g(t)=o(-\log t)$ $(C,1)$. Now (8) is not included in (1), but is included in (2). Thus the necessity part is false if (2) is omitted, though (2) may be replaced by the weaker condition (8). Similar comments probably apply to the other theorems of the scale. *L. S. Bosanquet* (London).

Bochner, S., and Chandrasekharan, K. On the localization property for multiple Fourier series. Ann. of Math. (2) 49, 966–978 (1948).

Let $\sum a_{n_1\cdots n_k}e^{i(n_1x_1+\cdots+n_kx_k)}$ be the Fourier series of $f(x_1,\cdots,x_k)$ and let

$$S_R^\delta(x)=\sum_{\nu\le R}(1-\nu^2/R^2)^\delta a_{n_1\cdots n_k}e^{i(n_1x_1+\cdots+n_kx_k)},$$

where $\nu^2=n_1^2+\cdots+n_k^2$, be the (ν^2,δ) typical means of the series. Here $\delta_0=\frac{1}{2}(k-1)$ is a critical exponent, since for $\delta>\delta_0$ we have the Fejér-like behavior of the means (i.e., summability at the points of continuity and the uniform boundedness of the means if f is bounded). For $\delta\le\delta_0$ it is no longer so, but if $k=1$, then for $\delta=\delta_0\,(=0)$ we have at least localization. That the latter result is false in the case $k\ge2$ was first shown by Bochner [Trans. Amer. Math. Soc. 40, 175–207 (1936)]. The (ν^2,δ_0) means of an $f\in L$ can then diverge at a point in the neighborhood of which f vanishes. The main result of the present paper is as follows. Let

$$f(x,t)=\frac{\Gamma(k/2)}{(2\pi)^{k/2}}\int_\sigma f(x_1+t\xi_1,\cdots,x_k+t\xi_k)d\sigma$$

be the spherical average of f, where σ stands for

$$\xi_1^2+\cdots+\xi_k^2=1.$$

Let, for $p>0$,

$$f_p(x,t)=\frac{2}{B(p,\frac{1}{2}k)t^{2p+k-2}}\int_0^t(t^2-s^2)^{p-1}s^{k-1}f(x,s)ds.$$

Then $\int_0^t f_p^2(x,y)dy=o(t)$ for $t\to0$ implies $\int_0^R|S_R^\delta(x)|^2dR=o(R)$ for $R\to\infty$, provided $\delta=p+(k-1)/2$. *A. Zygmund.*

Chandrasekharan, K. On Fourier series in several variables. Ann. of Math. (2) 49, 991–1007 (1948).

The author continues his investigations of the spherical summation of multiple Fourier series

$$\sum a_{n_1\cdots n_k}e^{i(n_1x_1+\cdots+n_kx_k)}\sim f(x_1,\cdots,x_k).$$

In previous papers [see, e.g., Proc. Indian Acad. Sci., Sect. A. 24, 229–232 (1946); these Rev. 8, 263] he gave formulas expressing the typical means (ν^p,δ) of the series in terms of the spherical means $f_p(t)$ of the function f, and conversely. The indices δ and p there had to satisfy certain restrictions. Here f_p is given in terms of the (ν^p,δ) means for all $p\ge0$, $\delta\ge0$. As a result the author is able to obtain various theorems on the existence of $\lim_{t\to+0}f_p(t)$ on condition that the series is (ν^p,δ) summable. In particular, he considers strong summability and absolute summability of the Fourier series.

A. Zygmund (Chicago, Ill.).

Offord, A. C. Approximation to functions by trigonometric polynomials. II. Fund. Math. 35, 259–270 (1948).

[For the first part of the paper, see Duke Math. J. 6, 505–510 (1940); these Rev. 1, 328.] It is a well-known fact that the theory of interpolation is not immediately applicable to L-integrable functions $f(x)$, since for the latter the behavior of f in any set of measure zero is irrelevant while the interpolating polynomials are totally defined by the values of f in a certain denumerable set of points (fundamental points). There are ways of bypassing this difficulty, and one of them [see Marcinkiewicz and Zygmund, Fund. Math. 28, 131–166 (1937)] is as follows. For any integer $n>0$ we set $x_i=2\pi i/(2n+1)$, $i=0,1,\cdots,2n$, and consider the trigonometric polynomial $I_{n,u}(x,f)$ of order n which coincides with a given (integrable) function f at the points x_i+u. We then make n tend to $+\infty$ and ask for the behavior of $I_{n,u}(x,f)$, as a function of x, for almost every u. The author proves among others the following results. (1) Let $\Delta f=\frac{1}{2}\{f(x+t)+f(x-t)-2f(x)\}$; then $I_{n,u}(x,f)$ converges to $f(x)$ at almost every point (x,u) of the square (Q) $0\le x\le2\pi$, $0\le u\le2\pi$, provided one of the following two integrals is finite:

$$\int^{2\pi}t^{-2}dt\int_0^{2\pi}|\Delta f|^r dx,\quad \int_0^{2\pi}t^{-(1+1/r)}dt\left(\int_0^{2\pi}|\Delta f|^r dx\right)^{1/r},$$

where $r>1$. (2) If $\int_0^{2\pi}t^{-1}dt\int_0^{2\pi}|\Delta f|^r dx<\infty$ for some $r>1$, then

$$\lim_{N\to\infty}N^{-1}\sum_{n=1}^N|I_{n,u}(x,f)-f(x)|=0$$

almost everywhere in Q. (3) Under the assumptions of (2), $I_{n_k,u}(x,f)$ converges to f almost everywhere in Q, provided $\{n_k\}$ satisfies a condition $n_{k+1}/n_k>q>1$ for all k. (4) Under the assumptions of (2), and if n_k is the same as before, both series

$$\sum_1^\infty n^{-1}|I_{n,u}(x,f)-I_{n,u}(x,f_n)|^r,\quad \sum_1^\infty|I_{n_k,u}(x,f)-I_{n_k,u}(x,f_{n_k})|^r$$

converge almost everywhere in Q, where

$$f_n(x)=(2\Delta_n)^{-1}\int_{x-\Delta_n}^{x+\Delta_n}f(t)dt,\quad \Delta_n=2\pi/(2n+1).$$

It is also shown that the assumptions of (1) are satisfied if f is a fractional integral of order r^{-1} of a $g\in L^r$.

A. Zygmund (Chicago, Ill.).

Wilkins, J. Ernest, Jr. Neumann series of Bessel functions. Trans. Amer. Math. Soc. **64**, 359–385 (1948).

The author considers an infinite series of Bessel functions of the first kind and of real index. From a classical integral he concludes that these Bessel functions form an orthonormal set in the interval between zero and infinity of the independent variable, with a weight function equal to the reciprocal of this variable. Whereas previous results were restricted to series of functions of index starting at unity, the author considers series in which this index starts at an arbitrary real number. For these cases he sets out to weaken Titchmarsh's integrability conditions regarding functions of a real variable. The proofs of the corresponding theorems are dependent upon a number of results on Bessel functions. These results are connected with the summation of a particular form of Neumann series.

The author proves a number of lemmas connected with the integral properties of Bessel functions. With the aid of these lemmas he is able to prove the formula for the expansion coefficients of a given function according to a series of Bessel functions as stated above. This theorem is extended for general values of the initial index. In order to prove a further expansion formula for a given function, the author starts from an elementary Hankel formula. He then states some additional results and finally considers a combination of two expansions of above type. *M. J. O. Strutt.*

Babenko, K. I. On conjugate functions. Doklady Akad. Nauk SSSR (N.S.) **62**, 157–160 (1948). (Russian)

Let $f(x)$ be of period 2π and of the class L, and let $\tilde{f}(x)=-\pi^{-1}\int_{-\pi}^{\pi}f(x+t)\frac{1}{2}\cot\frac{1}{2}t\,dt$ be the conjugate function of f. The author states without proof that (1) if $p>1$, $-1<\alpha<p-1$, then

$$(*)\qquad \int_{-\pi}^{\pi}|\tilde{f}|^p|x|^\alpha dx\leqq A_{p,\alpha}\int_{-\pi}^{\pi}|f|^p|x|^\alpha dx,$$

provided the right side here is finite (for $\alpha=0$ this reduces to M. Riesz's well-known inequality [Math. Z. **27**, 218–277 (1927)]; (2) the inequality (*) holds if $\tilde{f}$ is replaced by $f^*(x)$, with

$$f^*(x)=\sup_{0<\lambda\leqq\pi}\left|\pi^{-1}\int_{\lambda\leqq|t|\leqq\pi}f(x+t)\tfrac{1}{2}\cot\tfrac{1}{2}t\,dt\right|.$$

(3) It also holds for the Hilbert transform

$$\pi^{-1}\int_{-\infty}^{\infty}(x-t)^{-1}f(t)dt$$

of (nonperiodic) $f(x)$ defined in $(-\infty,\infty)$. (4) If

$$|f(x)|^p|x|^{p-2}\in L(-\infty,\infty),\qquad p\geqq 2,$$

if $F(x)$ is the Fourier transform of $f(x)$, and

$$f(x,a)=(2\pi)^{-1}\int_{-a}^{a}F(t)e^{-ixt}dt,$$

then

$$\lim_{a\to\infty}\int_{-\infty}^{\infty}|f(x)-f(x,a)|^p|x|^{p-2}dx=0.$$

(5) A system of elements $\{f_n\}$ in a Hilbert space H is called a basis if every $f\in H$ can be uniquely represented in the form $f=\sum a_kf_k$. N. Bary [C. R. (Doklady) Acad. Sci. URSS (N.S.) **54**, 379–382 (1946); these Rev. **8**, 513] considered a class of bases such that (†) $m(\sum a_k^2)^{\frac{1}{2}}\leqq\|\sum a_kf_k\|\leqq M(\sum a_k^2)^{\frac{1}{2}}$ for every $f\in H$, with m, M positive and depending on $\{f_k\}$ only. She also asked if (†) is a necessary condition for $\{f_k\}$ to be a basis. Using result (1) the author shows that it is not so. [In the case of even functions, theorem (3) has been proved by Hardy and Littlewood, cf. Duke Math. J. **2**, 354–382 (1936), especially theorem 12.] *A. Zygmund.*

Graffi, Dario. Un teorema sugli operatori funzionali e applicazione al teorema di reciprocità per le correnti variabili. Mem. Accad. Sci. Ist. Bologna. Cl. Sci. Fis. (10) **3**, 129–132 (1947).

[The nomenclature and notations of the author's memoir in Mem. Accad. Pont. Nuovi Lincei (3) **2**, 211–262 (1935) are used.] In the present paper the author proves that if an operator of class zero operating on $e^{i\omega_0 t}$ times the Heaviside unit function produces $\varphi(t)$, then the same operator operating on a function $V(t)$ of class zero produces

$$\left(\frac{d}{dt}-i\omega_0\right)\int_0^t V(t-\tau)\varphi(\tau)d\tau.$$

There is an application to the theory of electrical networks. [Cf. also the author's paper, Ann. Mat. Pura Appl. (4) **25**, 267–276 (1946); these Rev. **9**, 319.] [Reviewer's remark: Translated into the language of the Laplace transformation the result turns out to be a simple application of the product theorem.] *A. Erdélyi* (Edinburgh).

Pollard, Harry. Integral transforms. II. Ann. of Math. (2) **49**, 956–965 (1948).

A partial generalization is obtained of results of D. V. Widder [Duke Math. J. **14**, 217–249 (1947); these Rev. **9**, 89]. Let $0<a_1\leqq a_2\leqq\cdots$ be a sequence of real constants such that $\sum_{n=1}^{\infty}a_n^{-2}<\infty$ and such that, if $1/k(z)=\prod_1^{\infty}(1-z^2/a_n^2)$, then $k(z)=O(e^{-\beta|y|})$, $|y|\to\infty$, for a given $\beta>0$, uniformly for x in any finite interval. Let $K(s)=(2\pi i)^{-1}\int_{-i\infty}^{i\infty}e^{sz}k(z)dz$. The author's main result is that if $\int_{-\infty}^{\infty}K(s-t)\phi(t)dt=f(s)$ converges for a single real value of s then it converges for all real values of s and is inverted by the formula

$$\lim_{N\to\infty}\prod_1^N(1-D^2/a_n^2)f(t)=\phi(t),$$

which is valid for all t in the Lebesgue set of $\phi(t)$. This result includes as a special case the general convergence behaviour and inversion theory of the Stieltjes transform and its iterates. In addition to a number of typographical slips the following errors occur: the second moment of $\alpha_n(t)$ in § 2 is $2/a_n^2$; the limit in formula (5.1) is 0 not 2, a fact which makes possible certain simplifications in the preceding arguments. *I. I. Hirschman, Jr.* (Cambridge, Mass.).

Schwartz, Laurent. Sur une propriété de synthèse spectrale dans les groupes non compacts. C. R. Acad. Sci. Paris **227**, 424–426 (1948).

The author exhibits for $n\geqq 3$ a closed subspace A of L^1 over Euclidean n-space which is invariant under translation, and such that if B is the set of points at which all the Fourier transforms of functions of A vanish, then the subspace of all functions of L^1 whose Fourier transforms vanish on B is larger than A. In other words, using the terminology of Segal [Trans. Amer. Math. Soc. **61**, 69–105 (1947); these Rev. **8**, 438 (1947)] he exhibits a closed ideal in the group algebra of Euclidean n-space ($n\geqq 3$) which is not the kernel of its hull. Whether such ideals exist for $n=1$ or 2 is not known. *L. H. Loomis* (Cambridge, Mass.).

Tornehave, Hans. On a generalization of Kronecker's theorem. Danske Vid. Selsk. Mat.-Fys. Medd. **24**, no. 11, 21 pp. (1948).

A function H is called a phase-modulated oscillation if there exists a real c (mean motion of H) and a real almost periodic function $g(t)$ such that $H(t)=e^{i(ct+g(t))}$. The following is a generalization of Kronecker's theorem. If $H_1, \cdots, H_m$ have rationally independent mean motions, then to any real sequence $v_1, \cdots, v_m$ and $\epsilon>0$ there exists a t such that, for $\nu=1, \cdots, m$, $|H_\nu(t)-e^{iv_\nu}|<\epsilon$. The t satisfying this inequality are relatively dense. Another theorem asserts that any complex-valued almost periodic function f, bounded away from zero, can be written as $f=r\cdot H$, where r is positive almost periodic and H is a phase-modulated oscillation. Its mean motion is called the mean motion of f. A consequence of these two theorems is the following generalization of a result obtained by Jessen [Math. Ann. 111, 355–363 (1935)]. If $f_1, f_2, \cdots$ is a sequence of almost periodic functions bounded away from zero and α_i are complex numbers such that $\sum_{\nu=1}^{n}|\alpha_\nu|\sup_t|f_\nu(t)|<\infty$ and $\sum_{\nu=1}^{n}\alpha_\nu f_\nu=0$, then a finite subset of the mean motions of f_ν is rationally dependent. Finally this theorem can be extended even to relations of the form $Q_2(f_1, f_2, f_3)=k$, where Q_2 denotes a homogeneous quadratic form of its three arguments, and k a constant. *František Wolf* (Berkeley, Calif.).

Métral, Paul. Fonctions presque loxodromiques et fonctions presque automorphes. C. R. Acad. Sci. Paris **227**, 752–753 (1948).

A function $f(x)$ $(-\infty<x<\infty)$ is called almost loxodromic if it is continuous and to every $\epsilon>0$ there exists a number N such that in every interval (u, Nu) is situated a number a such that $|f(ax)-f(x)|<\epsilon$. A logarithmic transformation carries these functions over in the usual almost periodic functions. The mean value theorem and other theorems about almost periodic functions may be translated. The author states that a function $F(x)$ which is what he calls almost periodic associated with a continuous function $g(x)$ (i.e., to every $\epsilon>0$ there exists a number l such that every interval of length l contains a number τ such that $|F(x+\tau)-g(x)|<\epsilon$) possesses a mean value. The reviewer remarks that such a function $F(x)$ (as also $g(x)$) is almost periodic. A class of functions related to the almost loxodromic functions is also mentioned. *E. Følner.*

Polynomials, Polynomial Approximations

Walsh, J. L. On the location of the zeros of the derivatives of a polynomial symmetric in the origin. Bull. Amer. Math. Soc. **54**, 942–945 (1948).

In continuation of a previous paper [Mathematica, Cluj **8**, 185–190 (1934)], the author obtains some results on the zeros of the derivatives $p^{(k)}(z)$ of a polynomial $p(z)$ of degree n which is symmetric in the point $z=0$. Using the transformation $w=z^2$ and some previous results [Trans. Amer. Math. Soc. **22**, 101–116 (1921); **24**, 163–180 (1922)], he proves the following theorem. Let $p(z)$ have an m-fold zero at $z=0$ and all its remaining zeros in the closed interior of an equilateral hyperbola H with center at $z=0$. Let $p_0(z)$ be a $p(z)$ which has an m-fold zero at $z=0$ and its remaining zeros at the vertices of H, and let z_k denote a zero of $p_0^{(k)}(z)$ with the smallest positive modulus. Let H_k denote the equilateral hyperbola with center at $z=0$ and with a vertex at z_k. Then all the zeros $z\neq 0$ of $p^{(k)}(z)$ lie in the closed interior of H_k. A corresponding theorem resulting from the use of the transformation $w=z^q$ is also discussed. *M. Marden* (Milwaukee, Wis.).

v. Sz. Nagy, Gyula. Die Polarkreise eines Punktes in bezug auf ein Polynom. Portugaliae Math. **7**, 51–57 (1948).

Let $f(z)=(z-z_1)(z-z_2)\cdots(z-z_n)$ and $F(z)=\log f(z)$. The pth polar circle of $f(z)$ with center at z_0 is defined as the circle C_p: $|z-z_0|=r_p$ on which lie the p roots of the equation in t: $n/(z_0-t)^p=(-1)^{p-1}F^{(p)}(z_0)/(p-1)!$. It may be easily proved that at least one zero of $f(z)$ lies in or on each C_p and that, if no zero of $f(z)$ lies in a given C_p, all the zeros of $f(z)$ lie on that C_p. This theorem provides a "best possible" result. By computation of various r_p with $z_0=0$, various bounds may be derived for the zero of smallest modulus. Thus, if $f(z)=a_0+a_1z+\cdots+a_nz^n$ with $a_0\neq 0$, at least one zero of $f(z)$ is found to lie in each circle $|z|<r_p=|N_p/D_p|$ with $N_1=na_0$ and $D_1=a_1$; $N_2=n^1a_0$ and $D_2=(2a_0a_2-a_1^2)^1$; $N_3=n^1a_0$ and $D_3=(3a_0^2a_3-3a_0a_1a_2+a_1^3)^1$. For $f(z)=a_0+a_pz^p+a_{p+1}z^{p+1}+\cdots+a_nz^n$, it is found that $N_{p+h}=(na_0)^{1/(p+h)}$ and $D_{p+h}=[(p+h)a_{p+h}]^{1/(p+h)}$ for $h=0, 1, \cdots, p-1$; $N_{2p}=(na_0^2)^{1/(2p)}$ and $D_{2p}=[p(2a_0a_{2p}-a_p^2)]^{1/(2p)}$. Similar results are obtained for the polynomial

$$f(z)=a_0+a_pz^p+a_{p+1}z^{p+1}+a_{p+2}z^{p+2}+\cdots+a_nz^n.$$

The methods in this paper are similar to those in an earlier paper by the author [J. Reine Angew. Math. **169**, 186–192 (1933)]. *M. Marden* (Milwaukee, Wis.).

Conti, Roberto. Sul grado di approssimazione delle funzioni continue mediante polinomi di Stieltjes. Univ. Roma. Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) **7**, 91–102 (1948).

Let $f(x)$ belong to Lip α $(0<\alpha\leqq 1)$ in $0\leqq x\leqq 1$, and let $P_n(x)\equiv\frac{1}{2}k_n\int_0^1 f(u)[1-(u-x)^2]^n du$, $k_n^{-1}=\int_0^1(1-t^2)^n dt$. It is shown that in every interval $(\epsilon, 1-\epsilon)$, $0<\epsilon<\frac{1}{2}$,

$$|f(x)-P_n(x)|\leqq LKn^{-\alpha/2}+\{|f(0)|+|f(1)|\}n^{\frac{1}{2}}\epsilon^{-\epsilon^2n},$$

where L is the coefficient in the Lipschitz condition and K an absolute constant. Analogous results are obtained if $f'(x)\in$Lip α and for functions $f(x, y)$ of two variables. *A. Zygmund* (Chicago, Ill.).

Differential Equations

Neikirk, L. I. Some symbolic expansions. Univ. Washington Publ. Math. **2**, no. 3, 37–40 (1940).

Manipulations for the first order linear differential equation essentially equivalent to those used in the general Picard existence theorem [Picard, Traité d'Analyse, v. 2, Gauthier-Villars, Paris, 1893, p. 301]. *P. Franklin.*

Adamov, N. V. On certain transformations not changing the integral curves of a differential equation of the first order. Mat. Sbornik N.S. **23(65)**, 187–228 (1948). (Russian)

The basic problem of the paper is to determine whether or not the integral curve of $y'=f(x, y)$ through (a, y_0) remains in a given region $y_1(x)\leqq y\leqq y_2(x)$ for $a\leqq x\leqq b$. A transformation on the set G of continuous differentiable curves lying in this region is so defined that its fixed points are integral curves. Let $X(x, y)>0$ for curves in G, $Y(x, y)=Xf$, $F(x, y)=\int^y X dy$, and $A[y]=F+\int_a^x(Y-Xy')dx$. Then the

image of $y(x)$ under the transformation is $\bar{y}(x)$ where $F(x, \bar{y}) = A[y]$. The transformation is studied under various conditions on $\partial X/\partial x + \partial Y/\partial y$ so that one can try to show that G is mapped into a subset of itself.

Special applications are given which include a sufficient condition that the differential equation shall have a periodic solution, conditions on $p(x)$ to ensure that the solution of the Riccati equation $y' + y^2 + p(x) = 0$ through $(a, 0)$ shall remain positive for $x > a$, a condition on $p(x)$ (now periodic) that $u'' + p(x)u = 0$ shall have oscillating solutions, and a proof of the convexity of the space of continuous functions $p(x)$ of period ω such that $u'' + p(x)u = 0$ has nonoscillating solutions. *E. N. Gilbert* (Murray Hill, N. J.).

Šestakov, A. A. On the behavior of the integral curves of a system of ordinary differential equations in the neighborhood of a singular point. Doklady Akad. Nauk SSSR (N.S.) 62, 171–174 (1948). (Russian)

The equations

$$(1) \qquad \dot{x}_1 = \sum_{j=m}^{\infty} c_j x_1^j, \quad \dot{x}_i = \sum_{j=1}^{n} a_{ij} x_j + X_i(x_1, x_2, \cdots, x_n), \quad i = 2, \cdots, n,$$

are considered, $O: x_1 = \cdots = x_n = 0$ being an isolated singular point and the X_i being power series beginning with terms of at least the second degree. The characteristic roots λ_j of the matrix (a_{ij}), $i, j = 2, \cdots, n$, are supposed to have nonvanishing real parts. Theorem 1. If $\Re(\lambda_j) > 0$, $j = 1, \cdots, k-1$; $\Re(\lambda_j) < 0$, $j = k, \cdots, n$, and if $c_m > 0$, then given any system of k sufficiently small numbers $x_1^0, \cdots, x_k^0 (x_1^0 > 0)$, there is one and only one system $x_{k+1}^0, \cdots, x_n^0$ such that the solution passing through the point $(x_1^0, \cdots, x_n^0)$ tends to O as $t \to -\infty$. The singular point is classified as a node, a generalized saddle of the 1st, 2d or 3d type or a saddle-node, according to the signs of c_m and of the $\Re(\lambda_j)$ and whether m is even or odd. Theorem 2. If the λ_j are real and negative, $m \geqq 2$, the solutions tending to O are tangent at the origin to the curve defined by equating to zero the second members of the last $n-1$ equations (1). *J. L. Massera.*

Šestakov, A. A. The behavior of the integral curves of a system of the form

$$\frac{dx_1}{dt} = X_1(x_1), \quad \frac{dx_i}{dt} = \varphi_i(x_1, x_i) + X_i(x_1, x_2, \cdots, x_n)$$

in the neighborhood of a singular point. Doklady Akad. Nauk SSSR (N.S.) 62, 591–594 (1948). (Russian)

Previous results of the author [see the preceding review] are generalized to the present systems. He assumes that $O: x_1 = \cdots = x_n = 0$ is an isolated singular point, that $x_1 = 0$ is an isolated root of X_1, that $J = \int_0^{x_1} X_1^{-1} dx_1$ diverges, that $0 < m < |[\varphi_i(x_1, x_i) - \varphi_i(x_1, \bar{x}_i)]/(x_i - \bar{x}_i)| < M$ if $x_i \neq \bar{x}_i$ $(i = 2, \cdots, n)$ and that the X_i have continuous first partial derivatives vanishing at the origin. Theorem 2 of the previous paper is generalized as follows. In order that the integral curves which tend to O enter the origin along one and only one direction, the assumption $dX_1/dx_1 \to 0$ as $x_1 \to 0$ is sufficient. *J. L. Massera* (Montevideo).

Makarov, I. P. New criteria for stability according to Lyapunov in the case of an infinite triangular matrix. Doklady Akad. Nauk SSSR (N.S.) 62, 289–292 (1948). (Russian)

Stability in the sense of Lyapunov (with the norm $\sum |x_i|$) is investigated in the case of the system $\dot{x}_i = \sum_{k=1}^{\infty} p_{ik} x_k$ $(i = 1, 2, \cdots)$, where the p_{ik} are continuous functions of t. Theorem 1. If for $t > t_1 \geqq 1$ the following conditions are satisfied: (i) $p_{ii}(t) \leqq p_{ii}(t_1) < 0$; $|p_{ss}(t)| < |p_{ii}(t)|$ if $i < s$; $0 < |p_{ii}(t_1) - p_{ss}(t_1)| \leqq |p_{ii}(t) - p_{ss}(t)|$ if $i \neq s$; (ii) if $k < i$, then $p_{ik} \leqq t^{-(1+\beta_{ik})} \exp[\alpha_{ik}(t - t_1)]$, where $\alpha_{ik} \leqq |p_{ii}(t_1) - p_{kk}(t_1)|$ and the β_{ik} are positive constants such that the series $\sum_{i=1}^{\infty} 2^{k-2}/\beta_{ik}$ are uniformly bounded for all i; (iii) $p_{ik} = 0$ if $i < k$; then the solution $x_i = 0$ is asymptotically stable. Theorem 2. If assumption (ii) is replaced by: if $k < i$ the order of magnitude of $p_{ik}(t)$ is greater than that of $\exp \int_{t_1}^{t} |p_{ii}(t)|\, dt$ as $t \to \infty$, then the solution $x_i = 0$ is unstable. *J. L. Massera* (Montevideo).

Leonov, M. Ya. The parametric representation of quasi-harmonic oscillations. Doklady Akad. Nauk SSSR (N.S.) 62, 161–162 (1948). (Russian)

Solutions of the equation (1) $\ddot{x} + 2\gamma(\theta)\dot{x} + x = 0$ $(\dot{x} = dx/d\theta)$ are given in the form $x = \sin \varphi \exp(-2\int_{\theta_0}^{\theta} \gamma \cos^2 \varphi\, d\theta)$, where φ is the general solution of (2) $d\varphi/d\theta = 1 + \gamma \sin 2\varphi$, and c is a constant. The author introduces the function $\mu(\varphi) = \gamma\, d\theta/d\varphi$. The case when γ is periodic of period L, $\int_0^L \gamma\, d\theta > 0$, is considered. The author states that, for the existence of unstable solutions of (1): (i) it is necessary that a solution φ_p of (2) exists with the property $\varphi_p(\theta + L) = \varphi_p(\theta) + n\pi$, where n is an integer; (ii) if (i) is satisfied, it is necessary and sufficient that $|\int_0^{n\pi} \mu \cos 2\varphi\, d\varphi| > \int_0^{n\pi} \mu\, d\varphi$.

No proof is given of the first statement. The second one is apparently incorrect. *J. L. Massera* (Montevideo).

Demidovič, B. P. Periodic solutions of nonlinear systems of the second order of ordinary differential equations whose second members are periodic relatively to the independent variable. Doklady Akad. Nauk SSSR (N.S.) 61, 601–603 (1948). (Russian)

The author considers systems of the form

$$(1) \qquad \dot{x} = f_{10} + x f_{11} + y f_{12}, \quad \dot{y} = f_{20} + x f_{21} + y f_{22},$$

where the f are real continuous functions of x, y, t, periodic in t of period T, such that $f_{i0} = o(r)$ uniformly when $r^2 = x^2 + y^2 \to \infty$ and such that the solution of (1) through any (x_0, y_0, t_0) is unique and exists for $-\infty < t < +\infty$. Let (1′) be the "abbreviated" system obtained from (1) by dropping the terms f_{i0}. Lemma. Let $U = Ax^2 + 2Bxy + Cy^2$ be a positive definite form with constant coefficients whose derivative $\dot{U} = dU/dt$ (by virtue of (1′)) satisfies $\inf \dot{U}/r^2 > 0$ for $r \geqq r_0$. Then (1) has at least one periodic solution of period T.

The following theorems are consequences of the lemma. (I) Let $\lambda_i = \lambda_i(t, x, y)$ $(i = 1, 2)$ be the characteristic roots of the matrix (f_{ij}) $(i, j = 1, 2)$. Suppose that, for $r \geqq r_0$, (i) the real parts of the λ_i have the same and constant sign and $|\Re\lambda_1 + \Re\lambda_2| = |f_{11} + f_{22}| \geqq \alpha > 0$; (ii) $0 < |\lambda_1 - \lambda_2| < \beta |\lambda_1 + \lambda_2|$, where β is a constant which lies between 0 and 1 if the λ are real; (iii) $k_1 = \lim 2f_{12}/(f_{11} - f_{22})$ and $k_1 = \lim 2f_{21}/(f_{11} - f_{22})$ exist uniformly as $r \to \infty$; (iv) $k_1 k_2 + 1 \neq 0$. Then (1) has at least one periodic solution of period T. (II) The same is true if assumptions (ii) and (iii) are replaced by: (ii′) $|f_{11} - f_{22}| \leqq \beta_1 |f_{11} + f_{22}|$, β_1 constant; $f_{11} f_{22} - f_{12} f_{21} \geqq \gamma |f_{11} + f_{22}|^{\delta}$, γ, δ constants, $\gamma > 0$, $1 \leqq \delta \leqq 2$; (iii′) constants k_1, k_2 exist such that, for $r \to \infty$, we have uniformly $f_{12} = k_1 d + o(|d|^{\delta/2})$, $f_{21} = k_2 d + o(|d|^{\delta/2})$, where $d = (f_{11} - f_{22})/2$.

The reviewer observes that the lemma, which the author deduces from Schauder's fixed point theorem, can be easily obtained from a theorem by Levinson [Ann. of Math. (2) 45, 723–737 (1944); these Rev. 6, 173]. No proofs are given. *J. L. Massera* (Montevideo).

Revuz, André. Sur l'intégrabilité du système différentiel $(dy_i/dx)=f_i(x, y_1, y_2, \cdots, y_n)$. C. R. Acad. Sci. Paris **227**, 666–667 (1948).

The f's satisfy in $R: a<x<b$, $c_i<y_i<d_i$ the following conditions: (i) each f is finite and for x constant continuous in the y's; (ii) there is a set E whose section by each $x=$constant is everywhere dense in R and from each point $M^0=(x^0, y_1^0, \cdots, y_n^0)$ of which there issues a continuous arc $U(M^0)$ with equations $y_i=u_i(x, M^0)$, the functions f_i being continuous at M^0 on this arc; (iii) $u_i(x, M^0)$ has a derivative integrable on a segment to the right of x^0, continuous at x^0 and equalling $f_i(x^0, y_1^0, \cdots, y_n^0)$ at x^0; (iv) all the functions $u_i'(x)$ are bounded by the same function $M(x)$ which is integrable on (a, b). Under these conditions, through each point of R there is in R a continuous curve satisfying the differential system mentioned in the title for $x^0 \leqq x$, except possibly at the points of a certain set. The author remarks that his results overlap those of A. Marchaud [Bull. Soc. Math. France **62**, 1–38 (1934)]. *J. M. Thomas.*

Rydbeck, Olof E. H. On the propagation of waves in an inhomogeneous medium. I. Trans. Chalmers Univ. Tech. Gothenburg [Chalmers Tekniska Högskolas Handlingar] no. **74**, 35 pp. (1948).

With a view to studying wave propagation through inhomogeneous media, the author considers the one-dimensional wave equation (1) $d^2\Pi/dx^2+k_0^2n^2\Pi=0$, in which the refractive index n depends on x only. Introducing the W.K.B. approximations $\Pi^{(1)}$, $\Pi^{(2)}=n^{-\frac{1}{2}}\exp[\mp j\{\int^x n(s)ds-\pi/4\}]$, the author seeks to obtain the general solution of (1) as $\Pi=a^{(1)}\Pi^{(1)}+a^{(2)}\Pi^{(2)}$, while at the same time

$$d\Pi/dx=a^{(1)}d\Pi^{(1)}/dx+a^{(2)}d\Pi^{(2)}/dx.$$

Then the coefficients a must satisfy a pair of simultaneous first-order differential equations. With $b^{(1)}=a^{(1)}e^{j\Gamma}$, $b^{(2)}=a^{(2)}e^{-j\Gamma}$, these equations become

$$(2)\qquad db^{(1)}/dx=Hb^{(2)}e^{-2j\theta}, \quad db^{(2)}/dx=Hb^{(1)}e^{2j\theta},$$

where $\theta=W+\Gamma-\pi/4$, $W=k_0\int^x n(s)ds-\pi/4$,

$$\Gamma=\int^x H(s)ds, \quad H(x)=Q(x)/2k_0n(x),$$
$$Q(x)=\tfrac{3}{4}(n'/n)^2-\tfrac{1}{2}n''/n.$$

By a process of iteration, the rigorous equations (2) are approximately solved for slowly varying functions n. The solution leads to an expression for the physical reflection coefficient at any x. Certain known examples that permit exact solutions are treated in detail to show the usefulness of the author's method. Considerable attention is paid to cases where $n(x)$ possesses zeros. Physical applications concern magneto-hydrodynamic waves in the sun, and the propagation of microwaves in the lower troposphere.

C. J. Bouwkamp (Eindhoven).

Brillouin, L. A practical method for solving Hill's equation. Quart. Appl. Math. **6**, 167–178 (1948).

The author starts from a differential equation of the second order of linear homogeneous type, in which the dependent variable is multiplied by a real periodic function of the independent variable. He states that Floquet's theorem is satisfied by the solution of these equations. He then formulates the known solution by means of a Fourier expansion of the periodic coefficient, leading to Hill's determinant.

The author's solution is based on the direct application of Floquet's theorem. By this application he obtains a simple expression for the characteristic exponent of Hill's equation. He then proceeds to apply this method to a number of simple cases in which the periodic function depends parabolically on the independent variable and also exponentially. In these cases the solution is straightforward and simple. He then states the approximate solution of Hill's equation in the case where the periodic coefficient exhibits only small variations about a large average value. The asymptotic solution for this case is related to the author's direct solution.

He then shows how further approximations may be obtained from the said asymptotic solution. As a special case he considers a periodic coefficient which is a delta function of the independent variable. In this case his solution yields straightforward and simple results. Finally, he gives a comparison between his solution and the classical one.

M. J. O. Strutt (Zurich).

Biernacki, M. Sur les zéros des intégrales de l'équation $x^{(3)}(t)+A(t)\cdot x(t)=0$. Ann. Soc. Polon. Math. **21**, 26–37 (1948).

The paper is given to the proof of the following theorem: if $A(t)$ is a continuous nonincreasing function, and $0\leqq A(t)\leqq M$, on an interval I (finite or infinite), the subintervals of I between consecutive zeros of any solution $x(t)$ of the equation $x^{(3)}+A(t)x(t)=0$ are, with at most three possible exceptions, of lengths exceeding $(2+3^{\frac{1}{2}})^{-1}M^{-\frac{1}{2}}$. If $x(t)$ is oscillatory as $t\to\infty$, the number of exceptions is at most two. *R. E. Langer* (Madison, Wis.).

Aquaro, Giovanni. Sull'integrazione dei sistemi di equazioni differenziali lineari ordinarie. Atti Accad. Naz. Lincei. Mem. Cl. Sci. Fis. Mat. Nat. (8) **1**, 243–274 (1948).

This paper is concerned with a general boundary value problem relating to systems of linear differential equations. In the special case in which the system consists of one equation in one dependent variable, the problem can be stated as follows. Let the functions $a_0(x), a_1(x), \cdots, a_n(x)$, $f(x)$ be defined in an interval $A: \alpha_1\leqq x\leqq\alpha_2$. Let the typical function $a_k(x)$ be of class C^k, let $a_n(x)$ be positive, and let $f(x)$ be continuous in A. Let $I: x_1\leqq x\leqq x_2$ be a subinterval of A. Then it is required to find a function $u(x)$ of class C^n, defined over I, which satisfies the equation $\sum_{k=0}^{n}a_k(x)u^{(k)}=f(x)$ and the n conditions

$$\int_{x_1}^{x_2}f_i(x)u(x)dx+\sum_{k=0}^{n-1}\{\alpha_{ki}u^{(k)}(x_1)+\beta_{ki}u^{(k)}(x_2)\}=\gamma_i, \qquad i=1, \cdots, n,$$

where the given functions $f_i(x)$ are continuous in I, and the α's, β's and γ's are given constants such that the matrix $\|\alpha_{ki}, \beta_{ki}\|$ is of rank n.

The author's procedure is to reduce the problem to that of solving an infinite system of integral equations of the Fischer-Riesz type, and then to make use of known properties of such systems of equations to obtain conditions which are necessary and sufficient for the existence of a solution of the problem. The results obtained are complicated, requiring the use of a large amount of intricate symbolism, and they cannot be stated adequately in a brief review.

L. A. MacColl (New York, N. Y.).

Peierls, R. Expansions in terms of sets of functions with complex eigenvalues. Proc. Cambridge Philos. Soc. **44**, 242–250 (1948).

The author considers the differential equation

$$\psi''+[W-V(x)]\psi=0$$

in the interval $0 \leqq x \leqq 1$ subject to the boundary conditions $\psi = 0$ $(x=0)$, $d\psi/dx = f\psi$ $(x=1)$, where f is complex. He then demonstrates the following theorems. The differential equation plus boundary conditions determine f as a function of W. He shows that f is an analytic function of W, that the derivative of f with respect to W is a continuous function of W. From this he obtains the result that the eigenvalues of the original problem form a denumerable sequence which has no point of convergence except at infinity. Finally, he gives a plausibility argument for completeness of the eigenfunctions of equation (1). *H. Feshbach.*

***Saltikov, N. Metode Integraljenja Parcijalnih Jednačina Prvog Reda sa Jednom Nepoznatom Funkcijom. [Methods of Integration of Partial Equations of the First Order With a Single Unknown].** Serbian Academy of Sciences, Belgrade, 1947. xv+749 pp. (Serbian)

A detailed treatment of the subject, mostly following the lines of historical development. The following is a list of chapter headings (in an abbreviated version). Definition and origin. Euler's contribution. Lagrange's investigations and later generalizations. Linear equations. Method of Lagrange and its generalization by Jacobi. Theory of characteristics. Second method of Jacobi. Systems. Intermediate integrals. Methods of Lie. Tangential transformations. Differential invariants. Canonical equations and their applications in dynamics. Use of separation of variables. Applications to celestial mechanics. *W. Feller.*

Mendes, M. Sur les fonctions définies par un système d'équations aux dérivées partielles. J. Math. Pures Appl. (9) **27**, 177–204 (1948).

Poincaré, dans sa thèse [Paris, 1879], s'était proposé d'étudier, pour une équation aux dérivées partielles du premier ordre, les cas exceptionnels où le théorème de Cauchy ne s'applique plus. Ces cas sont de deux sortes: ou bien la difficulté provient non de la forme de l'équation proposée mais de la manière dont est définie l'intégrale particulière que l'on étudie, ou bien elle est due à la forme de l'equation proposée. De là la division naturelle de son travail en deux parties. Mendes étend les résultats de Poincaré au cas d'un système d'équations du premier ordre à une fonction inconnue. Il prend soin d'utiliser les notations du mémoire de Poincaré, dont il suit également l'ordre; il renvoie d'ailleurs à ce mémoire lui-même pour le détail de certaines démonstrations. Il sera sans doute intéressant d'illustrer le travail de Mendes par quelques exemples précis, simples, comme ceux que Poincaré avait donnés.

M. Janet (Paris).

Lednev, N. A. A new method for the solution of partial differential equations. Mat. Sbornik N.S. **22(64)**, 205–266 (1948). (Russian)

The author states that the main purpose of the paper is to develop a new method for the solution of Cauchy's problem [part II of the paper]. In order to do this it is found necessary to study first the solution of analytic systems [part I]. Part I constitutes an extension of results due to S. Kowalevsky [J. Reine Angew. Math. **80**, 1–32 (1875)] and C. Riquier [Les Systèmes d'Équations aux Dérivées Partielles, Gauthier-Villars, Paris, 1909]. The results of part II are related to those of I. G. Petrovskii [Bull. Univ. Moscow. Sér. Int. **1**, no. 7 (1938)].

Part I deals with the system

$$(1)\qquad z^{(i)}_{\omega_{i1}\cdots\omega_{in}} = F_i(x_1, \cdots, x_n; z_k; z^{(k)}_{\omega_1\cdots\omega_n}), \qquad i=1, 2, \cdots, r,$$

where

$$z^{(i)}_{\omega_1\cdots\omega_n} = \frac{\partial^{\omega_1+\cdots+\omega_n} z_k}{\partial x_1^{\omega_1}\cdots\partial x_n^{\omega_n}},$$

under the following two conditions: (I) each partial derivative which appears as an argument of F_i satisfies the inequality $\sum_{\nu=1}^{n}\omega_\nu \leqq \sum_{\nu=1}^{n}\omega_{k\nu}$, and it is understood that none of the derivatives occurring on the left of (1) also appears as an argument of one of the F_i; (II) the functions F_i are defined and analytic in a certain neighborhood G of the point $P=(a_\nu; b_k; b^{(k)}_{\omega_1\cdots\omega_n})$, $\nu, k=1, \cdots, r$. Before stating the main theorem, certain preliminary considerations are needed. For each pair of integers (i, ν) for which $\omega_{i\nu}>0$, let $\varphi(x^{(0)}_{i\nu})$, $\varphi^{(1)}_{i\nu}(x)$, $\cdots$, $\varphi^{(\omega_{i\nu}-1)}_{i\nu}(x)$ be $\omega_{i\nu}$ arbitrary functions of $x=(x_1, \cdots, x_n)$, and construct the "matrix" with at most m rows

$$A_i(x) = \begin{Bmatrix} \varphi^{(0)}_{i1}(x), \varphi^{(1)}_{i1}(x), \cdots, \varphi^{(\omega_{i1}-1)}_{i1}(x) \\ \varphi^{(0)}_{i2}(x), \varphi^{(1)}_{i2}(x), \cdots, \varphi^{(\omega_{i2}-1)}_{i2}(x) \\ \cdots\cdots\cdots\cdots \\ \varphi^{(0)}_{in}(x), \varphi^{(1)}_{in}(x), \cdots, \varphi^{(\omega_{in}-1)}_{in}(x) \end{Bmatrix},$$

for $i=1, 2, \cdots, r$. A set of r matrices $A_1(x), \cdots, A_r(x)$ is called an integral element at the point P for the system (1) provided that the functions $\varphi^{(\omega)}_{i\nu}(x)$ satisfy the following conditions: (a) they are defined and analytic in a neighborhood N of the point $a=(a_1, \cdots, a_n)$, and for each set of values i, ν, ω, the function $\varphi^{(\omega)}_{i\nu}(x)$ does not depend on x_ν; (b) in the neighborhood N of a we have

$$\{\partial^p \varphi^{(q)}_{i\mu}(x)/\partial x_\nu^p\}_{x_\nu=a_\nu} = \{\partial^q \varphi^{(p)}_{i\nu}(x)/\partial x_\mu^q\}_{x_\mu=a_\mu},$$

for $i=1, 2, \cdots, r$; $\mu, \nu=1, 2, \cdots, n$; $q=0, 1, \cdots, \omega_{i\mu}-1$; $p=0, 1, \cdots, \omega_{i\nu}-1$, whenever the integers i, μ, ν, q, p satisfy $\omega_{i\mu}>0$, $\omega_{i\nu}>0$; (c) at the point a we have

$$\frac{\partial^{\omega_1+\cdots+\omega_n}\varphi^{(\omega)}_{i\nu}(a)}{\partial x_1^{\omega_1}\cdots\partial x_n^{\omega_n}} = b^{(i)}_{\omega_1\cdots\omega_{\nu-1}\omega_\nu+1\cdots\omega_n}$$

whenever $\omega_\nu=0$ and the integers $i, \nu, \mu, \omega, \omega_\mu$ $(\mu\neq\nu)$ occur in the set of values which define the point P. If $z_i=z_i(x)$, $x=(x_1, \cdots, x_n)$, $i=1, 2, \cdots, r$, is an analytic integral of (1) in a neighborhood M of a, $M\subset N$, and

$$\{\partial^\omega z_i(x)/\partial x_\nu^\omega\}_{x_\nu=a_\nu} = \varphi^{(\omega)}_{i\nu}(x)$$

for $x\in M$; $i=1, 2, \cdots, r$; $\nu=1, 2, \cdots, n$; $\omega=0, 1, \cdots, \omega_{i\nu}-1$, then the given integral $z(x)=(z_1(x), \cdots, z_r(x))$ of (1) is said to pass through the given integral element at P. The generalized Cauchy problem consists in finding an analytic integral of the system (1) passing through a given analytic integral element at P. Another concept needed is the following. Fix i and k and consider the set of all derivatives $z^{(k)}_{\omega_1\cdots\omega_n}$ which appear as arguments of F_i and for which $\sum_{\nu=1}^{n}\omega_\nu = \sum_{\nu=1}^{n}\omega_{k\nu}$. Let

$$A_{i\omega}(\xi) = \sum_{(\omega)}\left|\frac{\partial F_i}{\partial z^{(k)}_{\omega_1\cdots\omega_n}}(P)\cdot\frac{\xi_1^{\omega_1}\cdots\xi_n^{\omega_n}}{\xi_1^{\omega_{k1}}\cdots\xi_n^{\omega_{kn}}}\right|,$$

where $\omega=(\omega_1, \cdots, \omega_n)$ satisfies $\sum_{\nu=1}^{n}\omega_\nu = \sum_{\nu=1}^{n}\omega_{k\nu}$, and $\xi=(\xi_1, \cdots, \xi_n)$; and consider the r by r matrix $A_P(\xi)=(A_{i\omega}(\xi))$. This matrix is called the characteristic matrix at the point P for the system (1), and its characteristic roots $\chi_P^{(\nu)}(\xi)$ $(\nu=1, 2, \cdots, r)$ are called the characteristic functions at the point P for the system (1). The main result [theorem 2] of part I is the following. Let $A_i(x)=(\varphi^{(\omega)}_{i\nu}(x))$ $(i=1, 2, \cdots, r)$ be an integral element at the point $P=(a_\nu; b_k; b^{(k)}_{\omega_1\cdots\omega_n})$ for the system (1), and $\chi_P^{(\mu)}(\xi)$ $(\mu=1, 2, \cdots, r)$ the corresponding characteristic functions at P, and suppose further that for some set of num-

bers (possibly complex) $\xi_1, \cdots, \xi_n$ we have $|\chi_\mu^{(\omega)}(\xi)|<1$ $(\mu=1, 2, \cdots, r)$. Then there exists one and only one analytic integral $z_i=z_i(x)$ $(i=1, 2, \cdots, r)$ of the system (1) passing through the given integral element at P. The proof proceeds by first setting up a formal power series solution (obtained is the usual way from the given Cauchy data) and showing that convergence takes place in a suitable neighborhood of a. In order to test the hypothesis of the theorem in special cases, the author is interested in finding conditions which will guarantee that all the characteristic values χ_ν, $\nu=1, 2, \cdots, r$, of a complex r by r matrix $A=(a_{ik})$ are less than one in absolute value. In this connection, he quotes a result of S. Gerschgorin [Bull. [Izvestia] Acad. Sci. Leningrad 1931, 749–754] that if $\sum_{i=1}^{r}|a_{ik}|<1$ for $i=1, 2, \cdots, r$ then $|\chi_\nu|<1$, $\nu=1, 2, \cdots, r$; and also proves [theorem 1] that for any matrix $A=(a_{ik})$ it is true that $|\chi_\nu|\leqq(\sum_{i,k=1}^{r}|a_{ik}|^2)^{\frac{1}{2}}$, $\nu=1, 2, \cdots, r$ [this theorem was proved earlier by A. B. Farnell, Bull. Amer. Math. Soc. 50, 789–794 (1944); these Rev. 6, 113]. The author employs Gerschgorin's criterion to show that certain of Riquier's theorems about passive systems are included in theorem 2.

Part II deals with system (1) again, but now the analyticity assumptions about the functions F_i are dropped, and it is only assumed that each F_i is continuous with respect to $x=(x_1, \cdots, x_n)$ and has continuous first derivatives with respect to the other arguments z_k and $z_{\omega_1}^{(k)}\cdots_{\omega_n}$ in a neighborhood G of the point P whose coordinates are $x_\nu=a_\nu$, $z_k=b_k$, $z_{\omega_1}^{(k)}\cdots_{\omega_n}=b_{\omega_1}^{(k)}\cdots_{\omega_n}$. An integral element is defined as before, the functions $\varphi_\nu^{(\omega)}(x)$ being continuous in a neighborhood of $a=(a_1, \cdots, a_n)$, together with all the partial derivatives

$$\frac{\partial^{\omega_1+\cdots+\omega_n}\varphi_\nu^{(\omega)}(x)}{\partial x_1^{\omega_1}\cdots\partial x_n^{\omega_n}} \qquad (\omega_\nu=0),$$

for which the partial derivative $z_{\omega_1}^{(i)}\cdots_{\omega_n}$ $(\omega_\nu=\omega)$ appears in the system (1). An integral $z(x)=(z_1(x), \cdots, z_r(x))$ is defined as usual, but all partial derivatives $z_{\omega_1}^{(k)}\cdots_{\omega_n}$ appearing in (1) are required to be continuous. The generalized Cauchy problem is formulated as before. The characteristic matrix and the characteristic functions are defined similarly. Two other concepts are found useful. The weight of the system (1) at P is defined to be $\inf_\xi[\lim_{N\to\infty}\sum_{\nu=1}^{r}|\chi_P(\xi)|^N]$, where $\xi=(\xi_1, \cdots, \xi_n)$. Clearly, the weight at P has one of the values $0, 1, \cdots, r, \infty$, and theorem 2 of part I states that the uniqueness and existence theorem holds for an analytic system of weight zero at P. The quantity

$$\sup_{P\in G}\left(\inf_{\xi}\left[\lim_{N\to\infty}\sum_{\nu=1}^{r}|\chi_P(\xi)|^N\right]\right)$$

is called the weight of the system (1) over the neighborhood G. The system (1) is called normal with respect to an argument x_τ, $1\leqq\tau\leqq n$, if and only if for each partial derivative $z_{\omega_1}^{(k)}\cdots_{\omega_n}$ appearing in (1) we have $\omega_\tau\leqq\omega_{k\tau}$, $k=1, 2, \cdots, r$, and such an argument x_τ is called a normal argument of (1). The main question raised is: how far do the results of part I continue to hold when the analyticity assumptions there made are relaxed for some of the arguments x_i? It is found that as far as the existence of a solution of Cauchy's problem is concerned, the analyticity conditions are superfluous for sets of normal arguments of (1). An idea of the results may be had from theorem 6. Suppose that the functions F_i of (1) satisfy a certain generalized Lipschitz condition in a neighborhood G of P, and that the weight of (1) over G is zero. Suppose further that $x_1, \cdots, x_p$; $0\leqq p\leqq n$, are normal arguments of (1) and that the integral element $A_i(x)=(\varphi_\nu^{(\omega)}(x))$ $(i=1, 2, \cdots, r)$ is such that: (a) in a neighborhood of $a=(a_1, \cdots, a_n)$ the functions $\varphi_\nu^{(\omega)}(x)$ and all their partial derivatives appearing in (1) satisfy a generalized Lipschitz condition with respect to $x_1, \cdots, x_p$, (b) in the same neighborhood the same functions are analytic with respect to $x_{p+1}, \cdots, x_n$. Then there exists at least one integral $z(x)=(z_1(x), \cdots, z_r(x))$ of system (1) passing through the given integral element, and the functions $z_i(x)$ and all their derivatives occurring in (1) satisfy a generalized Lipschitz condition with respect to the arguments $x_1, \cdots, x_p$, and are analytic with respect to $x_{p+1}, \cdots, x_n$.

J. B. Diaz (Providence, R. I.).

Krzyżański, Miroslaw. Sur le problème de Dirichlet pour l'équation linéaire du type elliptique dans un domaine non borné. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **4**, 408–416 (1948).

Let D be a closed and unbounded domain with boundary FD, and let $\{D_n\}$ be a sequence of closed and bounded domains contained in D such that $D_i\subset D_{i+1}$ and $D=\lim_{n\to\infty}D_n$. Let $\Phi(P)$ be a function defined and continuous in D. Assume that the Dirichlet problem: u_n a solution of the elliptic equation

$$E(u)\equiv\sum_{i,k=1}^{n}a_{ij}\frac{\partial^2 u}{\partial x_i\partial x_j}+\sum_{j=1}^{n}b_j\frac{\partial u}{\partial x_j}+cu=f(P), \qquad P=P(x_1, \cdots, x_n),$$

interior to D_n and $u_n=\Phi(P)$ on the boundary of D_n is solvable for every D_n. Assume further that there exists a function $H(P, k)$ defined for each point P of D and for each value $0<k<k_0$ of the parameter k, having the properties (a) $H(P, k)>0$, (b) $E(H)<-\delta(k)H$ for some positive number $\delta(k)$, (c) $H(P, k_1)/H(P, k_1+k_2)\to 0$ as $P\to\infty$ if $k_1, k_2>0$ and $0<k_1+k_2<k_0$, (d) there exist constants M and $0<\bar{k}<k_0$ such that $|\Phi(P)|\leqq MH(P, \bar{k})$, $|f(P)|\leqq MH(P, \bar{k})$. If the above assumptions hold, the author shows that $u=\lim_{n\to\infty}u_n$ exists and is a solution of the Dirichlet problem for domain D with $u(P)=\Phi(P)$ on FD. The solution is unique in the class of functions $|u(P)|\leqq MH(P, \bar{k})$.

F. G. Dressel (Durham, N. C.).

Carrier, G. F., and Ehlers, F. E. On some singular solutions of the Tricomi equation. Quart. Appl. Math. **6**, 331–334 (1948).

Consider the system $\phi_x=\psi_y$, $\psi_x=-\phi_y/y$, whence

$$\psi_{yy}+y\psi_{xx}=0, \quad \phi_{xx}+(\phi_y/y)_y=0. \tag{1}$$

The first equation of (1) is Tricomi's equation. The authors, using the Fourier transform method, seek, in the elliptic half plane $y>0$, solutions which can be expressed by means of hypergeometric functions. They obtain, in particular, the singular solution

$$\phi=(x^2+s^2)^{\frac{1}{6}}F(-\tfrac{1}{6}, \tfrac{1}{6}; \tfrac{1}{2}; x^2/(x^2+s^2)), \quad s=\tfrac{2}{3}y^{\frac{3}{2}}.$$

L. Amerio (Milan).

Rubinšteĭn, L. I. On the initial velocity of the front of crystallization in the one-dimensional problem of Stefan. Doklady Akad. Nauk SSSR (N.S.) **62**, 753–756 (1948). (Russian)

The author's solution of Stefan's problem [same Doklady (N.S.) **58**, 217–220 (1947); these Rev. **9**, 287] is examined as $t\to 0$. When solid and liquid have nonzero temperatures at the interface (front of crystallization), the flux of heat there and the velocity of the interface have the form $v=kt^{-\frac{1}{2}}[1+v^*(t)]$, where $v(+0)=0$. An evaluation of the

In the last line of the review it was stated that the proof was not complete. The proof may be completed and the necessary steps are given at the end of another paper by the author [Canadian Math. J. 3, 256–264 (1951)].

constant k is given. Corresponding results obtained for cases of zero interface temperature of solid and/or liquid are more complicated. *R. E. Gaskell* (Ames, Iowa).

Functional Analysis, Ergodic Theory

Stone, M. H. The generalized Weierstrass approximation theorem. Math. Mag. **21**, 167–184, 237–254 (1948).

L'auteur donne un exposé didactique de la théorie moderne du théorème de Weierstrass et de ses généralisations, qui prend sa source dans un travail de l'auteur [Trans. Amer. Math. Soc. **41**, 375–481 (1937)]. Soit X un espace compact, $\mathfrak{X}$ un ensemble de fonctions numériques continues dans X; imposant à $\mathfrak{X}$ des conditions de plus en plus restrictives, l'auteur caractérise dans chaque cas les fonctions qui peuvent être approchées uniformément dans X par des fonctions de $\mathfrak{X}$; les trois cas considérés pour $\mathfrak{X}$ sont: (1) quels que soient f et g dans $\mathfrak{X}$, $\sup(f, g)$ et $\inf(f, g)$ sont dans $\mathfrak{X}$; (2) $\mathfrak{X}$ satisfait à la condition précédente et est en outre un espace vectoriel sur les réels; (3) $\mathfrak{X}$ est une algèbre sur les réels. Dans chacun des trois cas on obtient en particulier des conditions pour que toute fonction continue sur X puisse être approchée uniformément par des fonctions de $\mathfrak{X}$; dans le troisième cas, on retrouve ainsi la généralisation du théorème de Weierstrass due à l'auteur [loc. cit.] et dans le second un théorème de Kakutani [Ann. of Math. (2) **42**, 994–1024 (1941); ces Rev. **3**, 205]; le résultat correspondant au premier cas est nouveau. De ces théorèmes généraux, l'auteur déduit d'abord les caractérisations des "idéaux" fermés dans l'ensemble de toutes les fonctions continues sur X, considéré soit comme un ensemble réticulé ("lattice"), soit comme un espace vectoriel réticulé, soit comme une algèbre. Les résultats s'étendent facilement aux fonctions à valeurs complexes. Dans une dernière partie, l'auteur montre enfin comment on peut très aisément rattacher au théorème de Weierstrass généralisé le théorème de prolongement d'Urysohn (pour les espaces compacts), l'approximation d'une fonction $f(x, y)$ continue dans un produit de deux compacts par des sommes $\sum_{i=1}^{n} u_i(x)v_i(y)$, où les u_i et v_i sont continues, l'approximation par les polynomes trigonométriques, les polynomes de Laguerre et ceux d'Hermite, et enfin le théorème de Peter-Weyl sur l'approximation des fonctions continues sur un groupe compact par des combinaisons linéaires des coefficients des représentations linéaires irréductibles du groupe. *J. Dieudonné* (Nancy).

Lorentz, G. G. Über die Grenzwerte in Verbänden. Math. Z. **51**, 404–422 (1948).

The author defines $\overline{\lim}$, $\underline{\lim}$, order-convergence, and *-convergence in the usual way. He also sets

$$o\text{-}\underline{\lim}\, y_n = \wedge\Big(\bigvee_{k=1,2,\cdots} y_{n_k}\Big),$$

where the meet is taken over all strictly monotone increasing sequences of subscripts; $o\text{-}\overline{\lim}^{*}\, y_n$ is the smallest y such that every subsequence has a subsubsequence with $\overline{\lim} \leqq y$; $u\text{-}\overline{\lim}$ and $u\text{-}\overline{\lim}^{*}$ are defined dually. Relationships between these various limits are studied; for instance, under a suitable distributivity condition, $u\text{-}\overline{\lim} = o\text{-}\overline{\lim}^{*} = \overline{\lim}$. Likewise the continuity of various limits is studied. In § 3 are considered metric lattices (defined by weaker postulates than usual) and criteria (in terms of their metric) for order- and *-convergence. This study is modified in § 4 by considering lattices in which $x \cap y$ may be the null element of the lattice (appropriately denoted by $-\infty$ in this paper), an example being the lattice of closed subsets of a compact metric space. *P. M. Whitman* (Silver Spring, Md.).

Lorentz, G. G. Operations in linear metric spaces. Duke Math. J. **15**, 755–761 (1948).

The principal result of this paper is the (partial) extension of the interior mapping theorem from spaces of type (F) to general linear metric spaces. Although it is known [cf. J. V. Wehausen, same J. **4**, 157–169 (1938)] and proved again by the author that a linear metric space may be remetrized with an equivalent F-metric, it is unknown whether completeness in the original metric implies completeness in the F-metric. Thus theorems involving only topological invariants such as category extend automatically to the general case, while the interior mapping theorem involves completeness of the domain and has a nonformal extension. However, the proof given of the latter is incomplete in that it strictly applies only to the case of a one-to-one mapping. *W. F. Eberlein* (Madison, Wis.).

Köthe, G. Die Stufenräume, eine einfache Klasse linearer vollkommener Räume. Math. Z. **51**, 317–345 (1948).

If μ is a linear set of sequences of complex numbers, let μ^* be the set of sequences $\{u_i\}$ for which $\sum_i |u_i a_i| < \infty$ for all $\{a_i\}$ in μ. [In the review of the preceding paper of this series, same Z. **51**, 17–35 (1947); these Rev. **9**, 358, the absolute value signs were, mistakenly, omitted.] The set μ is called perfect if $\mu^{**} = \mu$. If E is a countable set of sequences, the smallest perfect space $\mu = \mu(E)$ containing E is called the terrace space determined by E and μ^* is called an enterraced space. For example, the terrace space generated by the sequences $a^k = (1, k, k^2, \cdots)$, $k = 1, 2, \cdots$, is the space of coefficient sequences of power series convergent in a neighborhood of the origin, and the dual enterraced space is the space of coefficient sequences of entire functions. The author continues his studies of solutions of equations and of various topologies in perfect spaces by showing that many topological properties which may be distinct in general perfect spaces coincide in terrace or enterraced spaces. For example, there is given a large group of equivalent conditions on a matrix A mapping a terrace space μ into another μ'; these conditions assert various kinds of continuous invertability of A or its dual A', and various closure properties of $A(\mu)$ and $A'(\mu'^{*})$. Applications include Eidelheit's theorem [Studia Math. **6**, 139–148 (1936), p. 145] on solution of infinite systems of linear equations, and examples of subspaces with no complementary subspaces [Hagemann, Math. Ann. **114**, 126–143 (1937)]. *M. M. Day.*

Gagaev, B. On convergence in Banach spaces. Uspehi Matem. Nauk (N.S.) **3**, no. 5(27), 171–173 (1948). (Russian)

Let E be a Hausdorff space and let Φ be a set of continuous functions φ on E such that each $\varphi(E)$ is a Hausdorff space and $x = y$ if and only if $\varphi(x) = \varphi(y)$ for all φ in Φ. If $\{x_n\}$ is a sequence contained in E such that $\lim_n \varphi(x_n)$ exists for every φ in Φ, it is immediate that $\lim_n x_n$ exists if and only if every subsequence of $\{x_n\}$ has a convergent subsequence. This note states two special cases of this remark in which (a) E and E' are Banach spaces and either Φ contains a

single one-to-one function from E into E' or else Φ is the set of all linear mappings from E into E', and (b) E is a function space and Φ is the set of coefficient functionals in the expansion in terms of an orthonormal basis.

M. M. Day (Princeton, N. J.).

Ėzrohi, I. A. On the linear dimension. Doklady Akad. Nauk SSSR (N.S.) **62**, 35–38 (1948). (Russian)

Banach and Mazur [Studia Math. **4**, 100–112 (1933)] gave an example of two nonisomorphic Banach spaces of equal linear dimension [Banach, Théorie des Opérations Linéaires, Warsaw, 1932, p. 193]. This note gives an example of a space of the same linear dimension as, but not isometric to, its second conjugate. For $p>1$ let $T^p(Y)$ be the l_p product [Banach, p. 243] of the spaces $Y, Y^{(1)}, \cdots, Y^{(n)}, \cdots$, where $Y^{(1)}$ is the conjugate of Y and $Y^{(n+1)}=(Y^{(n)})^{(1)}$. Then [Day, Bull. Amer. Math. Soc. **47**, 313–317 (1941); these Rev. **2**, 221] for $p^{-1}+q^{-1}=1$, $[T^p(Y)]^{(1)}=T^q(Y^{(1)})$. Hence $[T^p(Y)]^{(2)}=T^p(Y^{(2)})$ is of the same linear dimension as $T^p(Y)$ for every Y. That $T^p(L)$ is not isomorphic to $T^p(L^{(2)})$, where L is the space of summable functions on $(0, 1)$, follows from the fact that L is not isomorphic to a conjugate space [Gel'fand, Mat. Sbornik N.S. **4**(**46**), 235–284 (1938)], and two facts from the author's thesis. The first characterizes Y as isomorphic to a conjugate space if and only if Y is boundedly weakly complete with respect to a total linear subset of $Y^{(1)}$. From this it follows that if $T^p(Y)$ is isomorphic to a conjugate space, so is Y. *M. M. Day.*

Schatten, Robert, and von Neumann, John. The cross-space of linear transformations. III. Ann. of Math. (2) **49**, 557–582 (1948).

[For part II see same Ann. (2) **47**, 608–630 (1946); these Rev. **8**, 31.] A cross norm α on the set $\mathfrak{B}_1\odot\mathfrak{B}_2$ of expressions $\sum_{j=1}^{n} f_j\otimes g_j$ is said to be uniform if

$$\alpha(\sum_{j=1}^{n} Sf_j\otimes Tg_j)\leqq|||S|||\cdot|||T|||\alpha(\sum_{j=1}^{n} f_j\otimes g_j)$$

for any pair of linear transformations S and T on $\mathfrak{B}_1$ and $\mathfrak{B}_2$, respectively. The greatest cross norm γ and least λ and the self-associate cross norm σ are uniform. Let A denote a transformation from $\mathfrak{B}_1$ to $\mathfrak{B}_2^*$ and let $(Af)g$ denote the result of applying Af to g. Then A is said to be of finite norm if there is a constant C such that for every expression $\sum_{j=1}^{n} f_j\otimes g_j$ we have $|\sum A f_j(g_j)|\leqq C\alpha(\sum f_j\otimes g_j)$ and the least such C is called the α norm of A, $|||A|||_\alpha$. In this paper, a Banach space $\mathfrak{B}$ of linear transformations A from $\mathfrak{B}_1$ into $\mathfrak{B}_2^*$ is called an ideal if (i) $A\in\mathfrak{B}$ implies $XAY\in\mathfrak{B}$ for X and Y linear and in $\mathfrak{B}_2^*$ and $\mathfrak{B}_1$, respectively, and (ii) $||XAY||\leqq|||X|||\cdot||A||\cdot|||Y|||$. It is shown that the set of transformations A from $\mathfrak{B}_1$ to $\mathfrak{B}_2^*$ of finite α norm is an ideal in this sense if and only if α is uniform. If $\mathfrak{B}_1$ and $\mathfrak{B}_2$ are unitary spaces $\mathfrak{H}_1$ and $\mathfrak{H}_2$ then a cross norm α is termed unitary invariant if $\alpha(\sum_{j=1}^{n} U\varphi_j\otimes V\psi_j)=\alpha(\sum_{j=1}^{n}\varphi_j\otimes\psi_j)$ for every unitary U and V on the appropriate space. It is shown that for unitary spaces uniformity and unitary invariance are equivalent. Furthermore, every unitarily invariant cross norm is reflexive. That is, it equals the associate of its associate. A final section discusses the process of completing the set of expressions $\sum f_j\otimes g_j$ by means of a unitarily invariant cross norm in the case when the resulting cross space corresponds to a set of completely continuous transformations. *F. J. Murray* (New York, N. Y.).

Kreĭn, M. G., and Rutman, M. A. Linear operators leaving invariant a cone in a Banach space. Uspehi Matem. Nauk (N.S.) **3**, no. 1(23), 3–95 (1948). (Russian)

This paper gives proofs and extensions of earlier work of the authors on the existence and properties of characteristic vectors and characteristic numbers of a monotone linear operator A in a partially ordered Banach space B. The various hypotheses of the theorems describe the nature of the cone K of nonnegative elements and impose conditions, such as complete continuity, on the operator A. The results, in particular § 6, extend such classical theorems as those of Frobenius and Perron on matrices with nonnegative elements, and the theorem of Jentzsch on integral operators with positive kernel. Section 9 has some results on certain nonlinear, completely continuous operators. The existence theorems are derived from the fixed point theorem for convex compact subsets of a locally convex linear topological space.

Section 1 deals with semi-groups in a Banach space B; that is, with convex sets K which are semi-groups under addition in B. The notation $x\leqq y$ means that $y-x\in K$; $x\ll y$ means that $y-x\in$(interior of K). Define K^* as the set of f such that $f\in B^*$ and $f(x)\geqq 0$ for all x in K. Call f in K^* positive (strictly positive) if there exists x in K (if for all $x\neq 0$ in K) $f(x)>0$. Typical preliminary results are as follows. Theorem 1.1. Let K be a cone, not all of B, and let G be a linear subspace of B containing an interior point of K; then every positive linear functional on G can be extended to a positive linear functional on B. Theorem 1.2. If K_1 is a semi-group with interior and K_2 a semi-group not containing an interior point of K_1, then there exists a positive f in K_1^* with $-f$ in K_2^*.

Section 2 introduces cones; that is, closed semi-groups K such that x and $-x$ both in K implies $x=0$. Call K normal if there exists $\delta>0$ such that $x_i\in K$ and $|x_i|=1$, $i=1, 2$, imply $|x_1+x_2|>\delta$. Call K minihedral if $\sup(x, y)$ and $\inf(x, y)$ exist for every x, y in K. If K has an interior, K^* is normal. If, moreover, K is minihedral, K^* is minihedral. If K has an interior point u, K is normal if and only if the interval $\{x|-u\leqq x\leqq u\}$ is contained in a sphere. In a finite-dimensional space a cone K is minihedral if and only if there exist linearly independent elements for which K is the set of nonnegative linear combinations of these elements. Kakutani's theorem on the representation of abstract M spaces can be rephrased as follows: a Banach space B is isomorphic to the space of continuous functions on a compact Hausdorff space if and only if there exists in B a normal, minihedral cone with interior.

Section 3 studies common fixed points and characteristic vectors of an Abelian family of operators and of their adjoints. (In sections 1–8 all operators are additive and continuous.) Theorem 3.3. Let K be a semi-group with interior and let G be a commuting family of linear transformations A with $AK\subseteq K$. Then there exists a positive φ in K^* which is a common characteristic vector of all the conjugate operators A^*, A in G; in fact, for each A in G there exists for each A a positive λ_A such that $A^*\varphi=\lambda_A\varphi$. Theorem 3.1. If in addition there is a common fixed point $u\gg 0$ of all A in G, then φ is a common fixed point of the A^*, A in G. This has as a corollary the existence of Banach limits for bounded real functions on Abelian groups.

Section 4 introduces the complex extension $\bar{B}$ of a given real Banach space B: $\bar{B}$ is the space of couples $z=x+iy$ of

elements of B with the usual norm

$$|\bar{z}| = \max_{0\leq\mu\leq 2\pi} |x\cos\mu + y\sin\mu|;$$

and the extension $\bar{A}$ of A: $\bar{A}\bar{z} = Ax + iAy$. The spectrum, resolvent set, and resolvent of A are defined to be the corresponding quantities for $\bar{A}$. Call K a reflexive cone if B is a reflexive space (so $K = K^{**}$) and both K and K^* have interiors. Theorem 4.1. Let K be a reflexive cone and A an operator mapping the interior of K into itself and such that $x \in K$ and $Ax = 0$ implies $x = 0$. Then there exist $x \neq 0$ in K, $f \neq 0$ in K^*, and a positive number ρ such that $Ax = \rho x$, $A^*f = \rho f$, and the spectrum S_A of A lies in the circle $|\lambda| \leqq \rho$.

These results are applied in § 5 to Lorentz mappings in Hilbert space H. Fix an orthogonal basis $\{e_j\}$ in H; for $x = \sum_{j=1}^{\infty} \xi_j e_j$, let $Jx = \xi_1 e_1 - \sum_2^{\infty} \xi_j e_j$. Then K, the set of x such that $(Jx, x) \geqq 0$, $(e_1, x) \geqq 0$, is a reflexive cone; in fact, $K^* = K$. If $(Ju, u) > 0$, then $[x, y] = -(Jx, y)$ is an inner product defining the original topology in the hyperplane $H_u = $ (set of all y such that $(Ju, y) = 0$). Call Γ a Lorentz mapping if it is a one-to-one function carrying H onto itself so that $(J\Gamma x, \Gamma y) = (Jx, y)$ for each x, y in H. It is shown that every Lorentz mapping is linear and either Γ or $-\Gamma$ maps K into itself. Theorem 5.1. Let Γ be a Lorentz mapping taking K into K; then Γ has a characteristic vector v in K and a characteristic number $\rho \geqq 1$ for which precisely one of the following two cases holds: either $\rho = 1$ and the whole spectrum of Γ is on the unit circle $|\lambda| = 1$, or else $\rho > 1$ and the spectrum of Γ consists of the simple points ρ and $1/\rho$, and of points on the unit circle. The structures of Lorentz mappings in case $\rho > 1$ (hyperbolic), and in case $\rho = 1$ but Γ has a fixed u interior to K (elliptic), are described. For example, in the elliptic case, H_u and the line through u reduce Γ, and Γ is a unitary transformation of H_u onto itself under the norm $[y, y]^{\frac{1}{2}}$. The remaining (parabolic) case is left to a later study by means of the motions (isometries) in the infinite dimensional Lobachevskyan space defined by remetrizing the set of all x such that $(Jx, x) = 1$ by $d(x, y) = \cosh^{-1}(Jx, y)$.

Section 6 begins the study of completely continuous operators which takes up the remainder of the paper: hereafter every cone K spans B (i.e., $K + (-K)$ is dense in B). Theorem 6.1. If the completely continuous linear operator A takes K into K and (a) has a point other than 0 in its spectrum, there exist a positive characteristic number ρ of A, not less in modulus than all other characteristic numbers of A, and vectors v in K and φ in K^* with $Av = \rho v$, and $A^*\varphi = \rho\varphi$. Theorem 6.2. If (a) of 6.1 is replaced by (a′) there is a u in K with $|u| = 1$, a $c > 0$, and an integer p such that $A^p u \geqq cu$, 6.1 holds with $\rho \geqq c^{1/p}$. Define A strongly positive to mean that for each x in K there is an integer n with $A^n x \gg 0$. Theorem 6.3. Let K be a cone with interior and let A be a completely continuous operator with $AK \subseteq K$. Then A is strongly positive if and only if (i) A has interior to K precisely one characteristic vector v, $Av = \rho v$, $\rho > 0$; (ii) the conjugate operator A^* has in K^* precisely one characteristic vector φ; this φ corresponds to the same ρ and is strictly positive; and (iii) the corresponding characteristic number ρ is simple and exceeds in modulus all other characteristic numbers of A.

Theorem 6.3 implies that a strongly positive A has a decomposition $Ax = \rho\varphi(x)v + A_1 x$, where $v \gg 0$, φ is strictly positive, $\varphi(v) = 1$, $Av = \rho v$, $A^*\varphi = \rho\varphi$, and the spectrum of A_1 is interior to the circle $|\lambda| = \rho$: that is, $\lim_n |A_1^n|^{1/n} < \rho$. Weaker decompositions are obtained in § 7 under weaker hypotheses. Theorem 7.2. If A is completely continuous and B has a new norm μ so that $\mu(Ax) \leqq \mu(x)$ for all x, then either $\lim_n |A^n|^{1/n} < 1$, or A has characteristic numbers of modulus 1 and $A = \sum_{j=1}^{k} \epsilon_j P_j + A_1$, where $\lim_n |A_1^n|^{1/n} < 1$, ϵ_j are distinct complex numbers of modulus 1, P_j are projections of B on finite-dimensional subspaces of B, $P_j P_k = \delta_{jk} P_j$, and $A_1 P_j = P_j A_1 = 0$. Theorem 7.3. If the μ of 7.2 is continuous, then $\mu(Px) \leqq \mu(x)$ for all x in H. If $U_n = \sum \epsilon_j^n P_j$, also $\mu(U_n x) = \mu(U_0 x) \leqq \mu(x)$. These results are applied to an A having a fixed vector v interior to a cone K by showing that $\mu(x) = \inf\{t \mid -tv \leqq x \leqq tv\}$ is a suitable μ, even continuous if K is normal.

In § 8 it is shown how these decompositions can be improved in case K is minihedral. Call U a permutator ($U\bar{H} \subseteq \bar{H}$) if U has a finite-dimensional range and in its range permutes some basis. Then U^* is also a permutator and the characteristic numbers of U are the zeros of $(\xi^a - 1)(\xi^b - 1) \cdots (\xi^p - 1)$, where $a, b, \cdots, p$ are the lengths of the cycles in the permutation associated with U. Say that $g_1, g_2, \cdots, g_n$ lie properly in K if $\sum_{j \leq n} c_j g_j \in K$ if and only if all c_j are nonnegative. Theorem 8.1. Let K be a minihedral cone and $AK \subseteq K$; if in addition A satisfies the conditions of 7.2, then to the conclusions of 7.2 may be added that $U = \sum \epsilon_j P_j$ is a permutator whose basis lies properly in K; hence the characteristic numbers of A of modulus 1 are roots of unity, and the set of fixed vectors of A is nonempty and has a basis lying properly in K. Theorem 8.2. If K is a minihedral closed cone with interior; if A is completely continuous, $AK \subseteq K$ and if A has a fixed vector interior to K, then the conclusions of 8.1 hold and the sets of fixed vectors of A and A^* have biorthogonal bases $v_1, \cdots, v_n$, $\varphi_1, \cdots, \varphi_n$ lying properly in K and K^* such that $\inf(\varphi_i, \varphi_j) = 0$ for $i \neq j$. These results are applied to stochastic n-by-n matrices ($a_{ij} \geqq 0$ and $\sum_{j \leq n} a_{ij} = 1$) and to integral equations in the space of continuous functions with stochastic kernels.

Section 9, related in some of its results to work of Rothe [Amer. J. Math. 66, 245–254 (1944); these Rev. 6, 71] considers certain nonlinear completely continuous monotone operators. For $c > 0$, A is called c-monotone with respect to K if it is monotone ($x \leqq y$ implies $Ax \leqq Ay$) and there exists u in K with $|u| = 1$ and $Au \geqq cu$. In addition, A is called rectilinear if $Ax = y$ implies $A(\lambda x) = \varphi_x(\lambda) y$, where $\varphi_x(\lambda)$ is real and nonnegative and $\varphi_x(0) = 0$. Theorem 9.3. Let A be a completely continuous rectilinear operator c-monotone with respect to K and suppose that for the u appearing in the definition of c-monotony we have $\lim_{\lambda \to 0} \varphi_u(\lambda) = \mu > 0$. Then A has a characteristic vector (indeed a characteristic ray) in K with positive characteristic numbers.

M. M. Day (Princeton, N. J.).

Cooper, J. L. B. One-parameter semigroups of isometric operators in Hilbert space. Ann. of Math. (2) **48**, 827–842 (1947).

A one-parameter semigroup has a set of isometric operators $U(t)$ defined for $t \geqq 0$ such that $U(0) = I$, $U(t) \cdot U(s) = U(t - s)$. Measurability is also assumed for most of the results quoted here. Let $\mathfrak{R}(t)$ denote the range of $U(t)$. Then $E(t) = U(t)U^*(t)$ is the projection on $\mathfrak{R}(t)$. If $E(t) \to E_\infty \neq 0$ as $t \to \infty$ then the space can be expressed as the orthogonal sum of two manifolds $\mathfrak{M}$ and $\mathfrak{M}_\infty$ which reduce the set $U(t)$ and on the latter the $U(t)$ behaves as that part of a Stone group of unitary transformations corresponding to $t \geqq 0$. Then $\mathfrak{M}$ in turn can be decomposed into a finite or infinite sum of manifolds $\mathfrak{M}_k$ reducing $U(t)$ and each of these is isomorphic

to $\mathfrak{L}_2(0, \infty)$ in such a way that $U(t)$ is the isometric transformation obtained by translating an amount t, i.e., $U(t)$ takes $f(x)$ into a function $g(x)$ with $g(x)=0$ for $x<t$ and $g(x)=f(x-t)$ for $x\geqq t$. This description of the structure of the semigroup is obtained by establishing the existence of an infinitesimal generator H such that $-i(d/dt)U(t)=HU(t)$. For a semigroup H is maximal symmetric. One obtains H by forming $A=\int_0^\infty U(t)e^{-t}dt$ and then the graph of H is the set of pairs $\{iAh, Ah-h\}$. *F. J. Murray.*

Gel'fand, I. M., Raĭkov, D. A., and Šilov, G. E. Commutative normed rings. Uspehi Matem. Nauk (N.S.) **1**, no. 2(12), 48–146 (1946). (Russian)

This is an exposition of the theory of commutative normed rings and its applications to topology and harmonic analysis as developed by the authors in various well-known papers [(1) Gelfand, Rec. Math. [Mat. Sbornik] N.S. **9(51)**, 3–24 (1941); these Rev. **3**, 51; (2) Gelfand, ibid., 51–66 (1941); these Rev. **3**, 51; (3) Gelfand and Šilov, ibid., 25–39 (1941); these Rev. **3**, 52; (4) Gelfand and Raikov, C. R. (Doklady) Acad. Sci. URSS (N.S.) **28**, 195–198 (1940); these Rev. **2**, 217; (5) Raikov, ibid., 296–300 (1940); these Rev. **2**, 223; (6) Šilov, ibid. **29**, 83–84 (1940); these Rev. **2**, 314; (7) Raikov, Trav. Inst. Math. Stekloff **14** (1945); these Rev. **8**, 133]. The treatment given of many topics is more complete than in the original articles with many more remarks and examples. Moreover some new results are presented. The authors report that their paper was actually written in 1940. The intervening literature is listed in a supplementary bibliography but is not taken account of in the body of the paper.

There are twenty-four sections. The first eleven are grouped together under the title "General theory of commutative normed rings." The remaining thirteen are distributed among four supplements with the titles: I, Certain topological questions in the theory of rings; II, Rings with convolution; III, The group ring of a locally compact group and the theory of characters; IV, The extension of maximal ideals.

Sections one through six and eight through eleven present the material of (1) in a somewhat rearranged and expanded form. Besides the theory of the radical leading up to the canonical homomorphism of a normed ring into the ring $C(\mathfrak{M})$ of all complex valued continuous functions on the compact Hausdorff space $\mathfrak{M}$ of all maximal ideals the chief topics discussed are, as in (1), real rings, analytic functions of the elements of a ring, direct sum decompositions, and the connection between the algebraic and topological structure of a ring.

A normed ring is said to be a ring with involution if it admits an involutary conjugate linear automorphism $x\to x'$ such that $e+xx'$ always has an inverse, e being the identity. In section 7 the following theorems about rings with involution are proved. For all x the canonical homomorphism takes $x+x'$ into a real valued function in $C(\mathfrak{M})$. A necessary and sufficient condition that a linear functional f on a normed ring be positive in the sense that $f(xx')\geqq 0$ for all x in the ring R is that there exist a Radon measure μ on $\mathfrak{M}$ such that for all x in R it is true that $f(x)=\int x(M)d\mu(M)$. It is a corollary that R has radical zero if and only if it has "sufficiently many" positive linear functionals.

The material in supplement I is substantially that in (3). A new topology is introduced into $\mathfrak{M}$ and the study of the resulting space when R is the ring $C(S)$ of all continuous functions on a topological space S leads to a theory of the compact extensions of S and their relationship to the closed subrings of $C(S)$. The connection between the new topology and the old is also treated.

Supplement II is like (2) in that it is devoted to the exploitation of the observation that Wiener's theorems on the inverses of functions defined by absolutely convergent Fourier series and integrals are consequences of the fact that an element of a normed ring has an inverse if and only if it belongs to no maximal ideal. The treatment here differs considerably from that given in (2) in the distribution of emphasis among the special rings considered. In particular the whole of section seventeen is devoted to a rather detailed study of the ring of functions of bounded variation on the real line with multiplication defined by the formula $f*g(x)=\int_{-\infty}^{\infty}f(x-y)dg(y)$. The authors do not succeed in giving an explicit determination of all of the maximal ideals in this ring. However, they discuss two important classes of such ideals and give a lengthy construction of an example of a maximal ideal which belongs to neither class. One of these classes has its elements in one-to-one correspondence with the real numbers in such a manner that f belongs to the ideal defined by λ if and only if $\int_{-\infty}^{\infty}\exp(i\lambda t)df(t)=0$. It is shown that if I is the intersection of all maximal ideals defined by real numbers in a certain closed interval then I is contained in no other maximal ideals than those used in its definition. This leads at once to a determination of the maximal ideals in the ring of all functions on a fixed finite interval which coincide there with some Fourier-Stieltjes transform. The section concludes with a necessary and sufficient condition that a function defined on a finite interval be extensible to a Fourier-Stieltjes transform. Among the results in the other section of supplement II which are not included in (2) is a generalization of a Mercerian theorem of Paley and Wiener [Paley and Wiener, Fourier Transforms in the Complex Domain, Amer. Math. Soc. Colloquium Publ., v. 19, New York, 1934, p. 59].

In supplement III certain basic facts about separable locally compact Abelian groups are derived from the theory of normed rings somewhat as in (4), (5) and (7). First the natural one-to-one correspondence between the maximal ideals of the group ring R of the group G and the one point compactification of its character group is set up. Then the theory of section seven is applied to demonstrate the existence of "sufficiently many" characters on G, the semi-simplicity of R, and the generalization to G of the Herglotz-Bochner positive-definite function representation theorem.

The final supplement is concerned with the following question. What maximal ideals of a normed ring R are extensible to maximal ideals in R' whenever R' is a normed ring containing R? A partial answer is given in terms of the "boundary" of the set $\mathfrak{M}$ of maximal ideals of R. It is shown that $\mathfrak{M}$ always admits a unique closed subset Γ which is minimal with respect to the property $\sup|x(M)|\ (M\in\Gamma)=\sup|x(M)|\ (M\in\mathfrak{M})$. Then Γ is called the boundary of $\mathfrak{M}$. The principal result [which includes the principal result of (6)] is that every $M\in\Gamma$ is extensible to R' for any $R'\supseteq R$. *G. W. Mackey.*

Šilov, G. E. On a property of rings of functions. Doklady Akad. Nauk SSSR (N.S.) **58**, 985–988 (1947). (Russian)

Let R be a complete normed ring consisting of functions defined on the interval $a\leqq t\leqq b$ (hereinafter specialized, without any loss of generality, to the case $a=0$, $b=2\pi$); D_n the class of all infinitely differentiable functions on that interval; and C_n the class of all functions with n continuous

derivatives there. The following theorem is proved: if R contains D_∞, then it contains C_P for some P. Previously published results [Gel'fand, Raĭkov, and Šilov, paper reviewed above; Šilov, Trav. Inst. Math. Stekloff 21 (1947); these Rev. 9, 596] permit comparison of R with normed rings built by introducing in D_∞ norms defined by the expressions

$$\max_{a\leq t\leq b} \sum_{n=0}^{\infty} |x^{(n)}(t)|\alpha_n/n!,$$

where $0\leqq\alpha_{m+n}\leqq\alpha_m\alpha_n$ and $\lim \alpha_n^{1/n}=0$, leading to the following basic result: if the function x in D_∞ has the Fourier coefficients a_n, where $2\pi a_n=\int_0^{2\pi}x(t)e^{int}dt$, then the relation $\lim_{n\to\pm\infty}\|a_ne^{int}\|=0$ holds in R. It then follows by a reductio ad absurdum that $\|e^{int}\|\leqq C(|n|^N+1)$ for some N, and hence that the required P exists. *M. H. Stone.*

Mikusiński, Jan G.-. L'anneau algébrique et ses aplications dans l'analyse fonctionnelle. I. Ann. Univ. Mariae Curie-Skłodowska. Sect. A. **2**, 1–48 (1947). (French. Polish summary)

The algebraic part of a previous note is given in more detail. Consider the class of functions which are summable on $0\leqq t<T$ where, if T is nonfinite, the existence of a Laplace transform is postulated. The M topology is used. With convolution as product this function class yields a ring. The author imbeds this ring in one obtained by the adjunction of the complex number field in the usual way and so obtains an identity element. Thus the rings considered are normed rings, though neither the fact nor the literature is cited in the paper. There are novelties in the approach to the application of the paper. This approach follows the classic developments of Volterra, but the explicit formulation of the ring situation makes the arguments more precise and more general. As is well known, the Laplace transform yields a similar formalism. The writer claims wider validity for the ring approach. However, his argument compares different function classes and a fairer comparison would be with the Laplace transform restricted to a finite range. *D. G. Bourgin* (Princeton, N. J.).

The statement in the review that a normed ring is considered is incorrect. Actually no topology is introduced in the paper.

Nemyckiĭ, V. V. On the theory of orbits of general dynamical systems. Mat. Sbornik N.S. **23**(65), 161–186 (1948). (Russian)

This paper contains proofs of results previously announced by the author [C. R. (Doklady) Acad. Sci. URSS (N.S.) **53**, 491–494 (1946); these Rev. **8**, 280].

W. H. Gottschalk (Philadelphia, Pa.).

Calculus of Variations

Giuliano, Landolino. Sulla continuità degli integrali curvilinei del calcolo delle variazioni. I. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **4**, 39–45 (1948).

A proof is given for the continuity of the integral

$$I(C)=\int M(x, y, z)dx+N(x, y, z)dy+P(x, y, z)dz$$

when the functions M, N, P and their partial derivatives N_x, N_z, P_x, P_y are continuous and $N_z=P_y$ in an open region of (x, y, z)-space. *L. M. Graves* (Chicago, Ill.).

Giuliano, Landolino. Sulla continuità degli integrali curvilinei del calcolo delle variazioni. II. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **4**, 183–189 (1948).

[See the preceding review.] A direct proof is given for the continuity of $I(C)$ on classes of absolutely continuous curves with bounded lengths, when the functions M, N, P are continuous on a closed region A of (x, y, z)-space. The author remarks that this result may also be obtained from preceding work of McShane [Ann. Scuola Norm. Super. Pisa (2) **3**, 183–211 (1934)]. It is also shown that for each sequence of absolutely continuous curves C_n which converges to an absolutely continuous curve C_0, and for which the sequence of lengths diverges to $+\infty$, a triple of continuous functions M, N, P may be constructed such that $I(C_0)\neq\lim I(C_n)$. *L. M. Graves* (Chicago, Ill.).

Cesari, Lamberto. Condizioni sufficienti per la semicontinuità degli integrali sopra una superficie in forma parametrica. Ann. Scuola Norm. Super. Pisa (2) **14** (1945), 47–79 (1948).

Let $F(x, y, z, u, v, w)$ be a continuous function of its variables such that $F(x, y, z, ku, kv, kw)=kF(x, y, z, u, v, w)$ for $k>0$. In a previous paper [same Ann. (2) **13** (1944), 77–117 (1948); these Rev. **9**, 505] the author has defined the integral $I(S)$ of F over any continuous surface S of the type of the 2-cell, assuming only that the Lebesgue area of S is finite. In the present paper, it is shown that $I(S)$ is a lower semi-continuous functional under very general assumptions concerning F. In particular, this is the case if F is semidefinite positive regular. The results generalize previous results of McShane and of the reviewer.

T. Radó (Columbus, Ohio).

Morrey, Charles B., Jr. The problem of Plateau on a Riemannian manifold. Ann. of Math. (2) **49**, 807–851 (1948).

The purpose of the paper is to solve a generalized version of the classical Plateau problem for the case when the prescribed boundary consists of a finite number of Jordan curves. The generalization consists of replacing Euclidean three-space by a Riemannian manifold of considerable generality. The general plan of approach is similar to that followed in the solution of the classical Plateau problem, and is most closely related to the work of Courant. However, a number of difficult issues arise. A detailed account cannot be given in view of the large number of concepts and definitions involved. A point of fundamental interest is the definition of surface area in a Riemannian manifold M. Since only arc length, and hence distance, is a priori given in M, the problem is to derive a concept of surface area from the concept of "distance." The "intrinsic area" of Busemann was set up to satisfy this need, but it lacks the property of lower semi-continuity essential for the direct method used in this paper. The author uses what may be termed an intrinsic version of the Lebesgue area. A more geometrical concept of this character may be a matter of interest. *T. Radó* (Columbus, Ohio).

TOPOLOGY

***Bourbaki, N. Éléments de mathématique. VIII. Première partie: Les structures fondamentales de l'analyse. Livre III: Topologie générale. Chapitre IX: Utilisation des nombres réels en topologie générale.** Actualités Sci. Ind., no. 1045. Hermann et Cie., Paris, 1948. ii+101+ii pp.

The title is an accurate indication of the point of view taken in this chapter. The following table of contents outlines the topics which are considered: (1) Écarts, (2) Metric and metrizable spaces, (3) Metrizable groups, absolute valued fields, normed spaces and algebras, (4) Normal spaces, (5) Baire spaces. There is also an appendix on infinite products in normed algebras. By making use of the material in the preceding chapters of this work, the main theorems in each of these topics are derived in a short space. Each section is followed by an excellent list of exercises.

E. G. Begle (New Haven, Conn.).

Aeschlimann, Florence. Sur les notions d'espace physique et de quasi-espace. C. R. Acad. Sci. Paris **227**, 179–181 (1948).

This deals with a type of abstract topological space which is intended to have applications in an abstract treatment of physical theories related to quantum mechanics. The author considers a space A together with a system F of subsets of A and a closure operation f defined only for the sets of the system F. Such a system (A, F, f) is called a quasi-space if no set of F is included in any other set of F. Two topologies are defined in a quasi-space, the first in terms of the system F only, the second involving also the closure operation f. If A has still a third topology when considered as a physical space, then this topology is said to respect the organization of A as a quasi-space (A, F, f) if closures of sets relative to it are always contained in the closures of the same sets as determined by the author's second topology.

O. Frink (State College, Pa.).

de Groot, J. Local connectedness and quasiorder. Nederl. Akad. Wetensch., Proc. **51**, 885–890 = Indagationes Math. **10**, 313–318 (1948).

In a connected compact metric space M a point is said to be of quasiorder less than or equal to n if it has small (not necessarily open) neighborhoods U, with $\dot{U} \cap \overline{M-U}$ having at most n components. The quasiorder of M is the supremum of the quasiorder of its points, and may be finite, ω, $\aleph_0$ or greater than $\aleph_0$. The results are as follows. The quasiorder of M is at most ω if and only if M is locally connected. If the quasiorder of M is at most $\aleph_0$, so also is the quasiorder of any continuous image of M. The notion of genus [K. Menger, Kurventheorie, Teubner, Leipzig-Berlin, 1932, p. 293] is weakened to quasigenus, and it is shown that the property of being of quasigenus at most α, α being an ordinal in the first or second class, is invariant under continuous maps. *J. L. Kelley* (Berkeley, Calif.).

van Dantzig, D. Divisibilité topologique d'un ensemble compact par un arc simple. Bull. Soc. Math. France **75**, 49–55 (1947).

Der kompakte separable Raum E sei doppelt-stetig in Bögen (topologische Bilder der abgeschlossenen Strecke) zerlegt, und diese Bögen seien so orientierbar, dass die Orientierung sich stetig durch E fortsetzt. Es wird bewiesen: E ist das Cartesische Produkt des Zerlegungsraumes und der Strecke. Der Verfasser erwähnt, dass er den Satz im Zusammenhang mit seinen Untersuchungen über die Grundlagen der Differentialgeometrie und über homogene Kontinuen 1926 gefunden hat. *H. Hopf* (Zürich).

Freudenthal, Hans. Sur un théorème topologique de M. van Dantzig. Bull. Soc. Math. France **75**, 56–62 (1947).

Der im vorstehenden Referat besprochene Satz von van Dantzig wird verallgemeinert und vollständig neu bewiesen. Der Raum E wird nicht als kompakt, sondern als separabel und metrisierbar vorausgesetzt; an die Stelle der Zerlegung von E in Bögen tritt die Voraussetzung, dass in E eine partielle Ordnung der Punkte gegeben ist und dass diese Ordnung gewisse Kompaktheits- und Zusammenhangs-Eigenschaften besitzt. Dann erweist sich für jeden Punkt $a \varepsilon E$ die Menge $T(a)$ derjenigen Punkte $x \varepsilon E$, welche im Sinne der partiellen Ordnung mit a vergleichbar sind, als homöomorph mit der Strecke, und für die Zerlegung von E in diese Bögen $T(a)$ gilt der Satz von van Dantzig.

H. Hopf (Zürich).

Fox, Ralph H. Homotopy groups and torus homotopy groups. Ann. of Math. (2) **49**, 471–510 (1948).

Im Rahmen der Homotopietheorie werden einem topologischen Raume Z neue Gruppen zugeordnet, Torus-Homotopiegruppen $\tau_n(Z)$ genannt $(n=1, 2, \cdots)$. Ihre Definition unterscheidet sich von derjenigen der Hurewiczschen Homotopiegruppen $\pi_n(Z)$ nur dadurch, dass man statt von Abbildungen einer Sphäre in Z von solchen eines Torus (eines topologischen Produktes mehrerer Kreislinien) ausgeht. Genauer: P sei ein Raum, $p \varepsilon P$ und $z \varepsilon Z$ je ein beliebiger, aber fester Punkt, Z^P der Raum aller Abbildungen f von P in Z mit $f(p)=z$, e die Abbildung $e(P)=z$; man betrachtet die Fundamentalgruppe $\mathfrak{F}$ von Z^P mit dem Basispunkt e. Ist hierbei P die $(n-1)$-Sphäre S^{n-1}, so ist $\mathfrak{F}=\pi_n(Z)$, ist P der $(n-1)$-Torus $T^{n-1}=S_1^1 \times S_2^1 \times \cdots \times S_{n-1}^1$, so ist $\mathfrak{F}$ die neue Gruppe $\tau_n(Z)$. Ihre Elemente können als Klassen von Abbildungen eines T^n in Z aufgefasst werden. Die Struktur von τ_n ändert sich nicht, wenn man den Punkt z durch einen andern ersetzt, der sich mit z durch einen Weg in Z verbinden lässt. Die Gruppe τ_1 ist die Fundamentalgruppe von Z; τ_n, $n \geqq 2$, ist im Gegensatz zu den π_n i.A. nicht Abelsch; τ_{n-1} ist einer Untergruppe von τ_n isomorph.

Der Zusammenhang zwischen den $\tau_n(Z)$ und den $\pi_n(Z)$ wird vollständig beschrieben, was z.T. nicht einfache Konstruktionen und Beweise erfordert. Für $n \leqq r$ lässt sich π_n isomorph in τ_r einbetten, und zwar auf $\binom{r-1}{n-1}$ verschiedene Arten: jeder dieser Isomorphismen I_n benützt nur $n-1$ der $r-1$ Kreisfaktoren S_i^1 von T^{r-1}. Ist $a \varepsilon \pi_n$, $b \varepsilon \pi_m$, $\bar{a}=I_n a \varepsilon \tau_r$ und $\bar{b}=I_m b \varepsilon \tau_r$, so ist der Kommutator $\bar{a}\bar{b}\bar{a}^{-1}\bar{b}^{-1}=I_{n+m-1}(aob)$ oder $=1$, je nachdem die Einbettungen I_n und I_m lauter verschiedene Kreisfaktoren S_i^1 benützen oder nicht; dabei bezeichnet I_{n+m-1} eine geeignete Einbettung von π_{n+m-1} in τ_r, und $aob \varepsilon \pi_{n+m-1}$ das J. H. C. Whiteheadsche Produkt von a und b [Ann. of Math. (2) **42**, 409–428 (1941); diese Rev. **2**, 323], welches als Spezialfall die von Eilenberg [Fund. Math. **32**, 167–175 (1939)] eingeführte Operation von π_1 in π_n umfasst. Die algebraische Struktur der Gruppen τ_r ist durch diejenige der π_n, $n \leqq r$, und das Whitehead-Produkt in den Dimensionen $\leqq r$ völlig bestimmt: τ_r enthält eine Untergruppe τ_r', die dem direkten Produkt von je $\binom{r-2}{i-2}$ Exemplaren π_i, $i=2, 3, \cdots, r$, isomorph ist, und τ_r ist eine Erweiterung von τ_r' durch τ_{r-1} mit trivialem Faktor-

system und mit Operationen von τ_{r-1} in τ_r', die dem Whitehead-Produkt entnommen werden können. Es wird ferner gezeigt, dass für $r>1$ die Homotopiegruppe ε_r von Abe [Jap. J. Math. 16, 169–176 (1940); diese Rev. 2, 71] einer Untergruppe von τ_r isomorph ist.

Die Gruppen τ_n gestatten eine einfache Formulierung der Sätze von J. H. C. Whitehead über das Verhalten der Homotopiegruppen beim Einspannen eines Elements. Es sei Z ein Polyeder, Z' entstehe aus Z durch Entfernen des Innern E eines r-dimensionalen Hauptsimplexes mit der Randsphäre s^{r-1}. Die Elemente von $\tau_n(Z')$, welche durch Abbildungen in s^{r-1} repräsentiert werden können, bilden eine Untergruppe $\tau_n^-(Z')$ von $\tau_n(Z')$. Beim Einspannen von E geht eine Untergruppe $\tau_n^+(Z')$ von $\tau_n(Z')$ in die Eins von $\tau_n(Z)$ über. Die Ergebnisse von Whitehead werden dahin zusammengefasst, dass für $n\leqq r$ die Gruppe $\tau_n^+(Z')$ der von $\tau_n^-(Z')$ erzeugte Normalteiler von $\tau_n(Z')$ ist.

Analog den relativen Homotopiegruppen $\pi_n(Z, Y)$ von Z bezüglich des Teilraumes Y, $n\geqq 2$, sind die relativen Torus-Homotopiegruppen $\tau_n(Z, Y)$ definiert; mit den oben verwendeten Bezeichnungen und mit $P=T^{n-2}$ kann $\tau_n(Z, Y)$ als $\pi_2(Z^P, Y^P)$ erklärt werden. Die relativen und absoluten τ_n bilden eine Homotopiesequenz, welche bis auf kleine Unterschiede derjenigen der π_n entspricht. Diese wird übrigens in der Arbeit übersichtlich und unabhängig von andern Darstellungen beschrieben und diskutiert, ebenso das Whitehead-Produkt und die Deutung der Homotopiegruppen in verschiedenen Abbildungsräumen. *B. Eckmann.*

Bing, R. H. A homogeneous indecomposable plane continuum. Duke Math. J. 15, 729–742 (1948).

This paper answers in the negative a long-standing question of Knaster and Kuratowski [Fund. Math. 1, 223 (1920), problème 2]. If a nondegenerate plane continuum is homogeneous, is it necessarily a simple closed curve? Mazurkiewicz has shown [Fund. Math. 5, 137–146 (1924)] that for nondegenerate plane Peano continua the answer is in the affirmative. Both Choquet [C. R. Acad. Sci. Paris 219, 542–544 (1944); these Rev. 7, 335] and Waraszkiewicz [C. R. Acad. Sci. Paris 204, 1388–1390 (1937)] have stated without proof that the answer to the question is in the affirmative. The reviewer finds Bing's argument convincing. The author's counter-example is a continuum which is homeomorphic to a continuum described for a different purpose by E. E. Moise [Trans. Amer. Math. Soc. 63, 581–594 (1948); these Rev. 10, 56]. It requires ten pages of argument to show that it actually has the stated property. *J. H. Roberts* (Durham, N. C.).

Whyburn, G. T. Sequence approximations to interior mappings. Ann. Soc. Polon. Math. 21, 147–152 (1948).

The spaces A and B being separable metric, continuous mappings $f(x)$ of A onto open subsets of B are studied with special reference to the relations between uniform convergence and interiority. It is known that the limit mapping of a uniformly convergent sequence of interior mappings may not be interior. It is shown that (1) for A locally compact, and sequence f_n converging to f uniformly on each compact subset of A, a necessary and sufficient condition that f be interior is that for any compact subset A_0 of A and $\epsilon>0$, there exist $\delta>0$ and N such that for $n>N$, a σ_n-neighborhood of $f_n[V_\epsilon(x)]$ contains the δ-neighborhood of $f_n(x)$ for all $x\varepsilon A_0$, where $\sigma_n\to 0$ with $1/n$.

A sequence f_n is called uniformly approximately interior on $A_0\subset A$ if $\epsilon>0$ implies the existence of $\delta>0$ and N such that for $n>N$ and $x\varepsilon A_0$, $f_n[V_\epsilon(x)]\supset V_\delta[f_n(x)]$. It is shown that (2) if the interior mappings $f_n(A)\subset B$ converge uniformly to the light mapping $f(x)$ on an open subset W of A, where A and B are locally compact and locally connected, then the sequence is uniformly approximately interior on any compact subset A_0 of W. Thus from (1) and (2) follows that, if A and B are locally compact and locally connected, and the interior mappings $f_n(A)$ converge to the light mapping $f(A)$, the convergence being uniform in each compact subset of A, then the sequence f_n is uniformly approximately interior on each compact subset of A and f is interior on A. *R. L. Wilder* (Ann Arbor, Mich.).

Aleksandrov, A. D. Geometry and topology in the Soviet Union. I. Uspehi Matem. Nauk (N.S.) 2, no. 4(20), 3–58 (1947). (Russian)

Aleksandrov, A. D. Geometry and topology in the Soviet Union. II. Uspehi Matem. Nauk (N.S.) 2, no. 5(21), 9–92 (1947). (Russian)

GEOMETRY

***Coxeter, H. S. M. Regular Polytopes.** Methuen & Co., Ltd., London, 1948; Pitman Publishing Corporation, New York, 1949. xix+321 pp. (8 plates). 50s. Great Britain; $10.00, U.S.A.

The study of the regular polyhedra, which was the culmination of Euclid's Elements, is continued here and extended to include the analogous shapes in four and more dimensions. The most recent books on the subject are by Brückner [Vielecke und Vielfache; Theorie und Geschichte, Leipzig, 1900; Über die gleicheckig-gleichflächigen discontinuerlichen und nichtkonvexen Polyeder, Halle, 1906], Schoute [Mehrdimensionale Geometrie, II. Teil, Die Polytope, Leipzig, 1905], Elte [Groningen thesis, 1912] and Sommerville [An Introduction to the Geometry of n Dimensions, Methuen, London, 1929]. This book is a more complete treatise. It includes some of the work omitted from the earlier books as well as the recent work, much of which is taken from the eighteen papers on the subject published by the author since 1928. An introduction is given to certain of his discoveries not yet published elsewhere.

The early chapters cover the older theorems about polygons, polyhedra and the higher polytopes in terms of modern topological methods and notation. For example, the Schläfli symbol $\{p, q\}$ is used for the regular polyhedron all of whose faces are p-gons, with q of these p-gons meeting at each vertex. The symbol N_i is used to denote the number of i-dimensional elements of the polyhedron. Relations between these quantities, such as $N_0=4p/[4-(p-2)(q-2)]$, $N_1=2pq[4-(p-2)(q-2)]$ and $N_2=4q/[4-(p-2)(q-2)]$, are derived. The possible polyhedra then follow as integral solutions of these Diophantine equations.

The process of reciprocation with respect to a sphere through the mid-points of the edges gives the reciprocal polyhedron. The region common to both polyhedra is called a quasi-regular polyhedron. The polyhedron which contains both is a rhombic polyhedron. This is a special case of a zonohedron which is bounded entirely by parallelograms. Many interesting relationships among these shapes are derived and the space-filling shapes are determined.

There is a short introduction to group theory, which is then applied to the determination of the rotation groups of the regular polyhedra. A simple example of one of the theorems is the one which states that the order of the group is twice the number of edges of the regular polyhedron.

Tesselations and honeycombs are then considered as special cases of filling the plane with regular polygons or space with regular polyhedra. A chapter is devoted to the kaleidoscope derivation of groups generated by reflections and their relation to the regular polyhedra and the honeycombs. Representation by graphs is employed to clarify the derivations. Star polyhedra are then derived by extending the faces to form nonconvex shapes.

The foregoing ideas are extended to the corresponding polytopes in four and more dimensions. The regular, quasi-regular and semi-regular polytopes, including the three-dimensional Platonic and Archimedean solids, are derived. A chapter on vectors and quadratic forms is included to explain the notation used for the higher dimensional cases produced by the generalized kaleidoscope.

The new material is principally the author's work on the generalized kaleidoscope derivation of reflection groups and their application to the derivation, description and enumeration of regular and semi-regular polytopes and honeycombs. The author's concise graphical notations, most of which resemble tree graphs, give the number and arrangement of the mirrors. It is shown that every group generated by reflections is a direct product of groups whose fundamental regions are simplexes: a spherical simplex in the case of finite groups and a Euclidean simplex in the case of an infinite irreducible group. The only irreducible discrete groups generated by reflections are those whose fundamental regions are included in a given list of twenty simplexes. This method is applied to the derivation of polytopes and to the computation of the number of the various geometrical elements of many polytopes including Gosset's semi-regular polytopes and honeycombs in six, seven and eight dimensions. It is shown that there are no regular star honeycombs in four and more dimensions. A beginning is made on the Goursat problem of determining the spherical tetrahedra which lead, by repeated reflection in their faces, to a finite set of congruent tetrahedra (a honeycomb covering the hypersphere a finite number of times). The number of solutions is expected to run into the hundreds.

In addition to the eight plates, many figures are used to clarify the complicated geometrical notions. Many of the numerical values of geometrical quantities associated with polytopes are given on fourteen pages of tables. A nine page bibliography lists the most important references. Historical remarks are included at the end of nearly every chapter.

M. Goldberg (Washington, D. C.).

van der Woude, W. On projective transformations of the binary quartic form. Simon Stevin **26**, 1–11 (1948). (Dutch)

This paper begins with the well-known fact that two binary quartic forms, $f=x_0t^4+4x_1st^3+6x_2s^2t^2+4x_3s^3t+x_4s^4$ and $f'=x_0't^4+\cdots$, are (in general) projectively equivalent if $I^3/J^2=I'^3/J'^2$, where $I=x_0x_4-4x_1x_3+3x_2^2$, $J=x_0x_2x_4+2x_1x_2x_3-x_0x_3^2-x_1^2x_4-x_2^3$, $I'=x_0'x_4'-\cdots$ and $J'=x_0'x_2'x_4'+\cdots$. The author asks what further conditions are needed in the critical cases (1) when $I^3/J^2=I'^3/J'^2=27$, (2) when $I=J=I'=J'=0$. He regards s and t as homogeneous parameters for a point $(s^4, -s^3t, s^2t^2, -st^3, t^4)$ on the rational normal quartic curve C in projective 4-space, so that $f=0$ is the condition for the osculating hyperplane at this point to pass through $(x_0, x_1, x_2, x_3, x_4)$. Such a point (x) of general position lies on four osculating hyperplanes, and their points of contact lie on the polar hyperplane of (x) with respect to the quadric $I=0$. The other invariant equation, $J=0$, represents the cubic hypersurface generated by the chords of C. The tangents of C generate a sextic surface R whose equations are $I=J=0$. Any two osculating planes of C intersect in a point, and the locus of such points is a quartic surface D. The conclusion is as follows. In case (1), f and f' are projectively equivalent if points (x) and (x') both lie on D but not on C. In case (2), they are projectively equivalent if (x) and (x') lie on R but not on C.

H. S. M. Coxeter (New York, N. Y.).

Chen, Shou. On the application of vector algebra to projective geometry. Amer. Math. Monthly **55**, 541–545 (1948).

Algebraic Geometry

***Weil, André. Sur les courbes algébriques et les variétés qui s'en déduisent.** Actualités Sci. Ind., no. 1041 = Publ. Inst. Math. Univ. Strasbourg **7** (1945). Hermann et Cie., Paris, 1948. iv+85 pp.

Suppose that Ω_k is a field of algebraic functions of one variable with a finite field k of q elements for a coefficient field. Let p be a prime divisor of Ω_k which is trivial on k, i.e., a homomorphism of a suitable subring of Ω_k upon an algebraic extension $k(p)$ of k with $[k(p):k]=d(p)$. Then the zeta function of Ω_k can be defined as $Z(u)=\prod_p(1-u^{d(p)})^{-1}$, where u is a complex variable, and the product is extended over all distinct prime divisors. Artin formulated the analogue of the Riemann hypothesis as the statement that the zeros of $Z(u)$ lie on the circle $|u|=q^{-1}$. The author gives in this paper the first complete proof of this Riemann hypothesis for function fields of arbitrary genus. His proof depends on a reformulation of the hypothesis as an assertion on the positiveness of a quadratic form [corollary 3 on page 70]. This quadratic form is derived from a trace function σ acting on a subring of the ring of correspondences of Ω_k. Thus the author requires a complete treatment of the theory of correspondences, and the major portion of the present paper is devoted to it.

For the unity of method and in order to establish a clear-cut connection with the original papers of Castelnuovo, Enriques, Severi and others, it is found convenient to discuss "curves Γ over k" instead of the function field Ω_k. The concept of "curve" as used by the author requires a careful explanation as given in his book "Foundations of Algebraic Geometry" [Amer. Math. Soc. Colloquium Publ., v. 29, New York, 1946; these Rev. **9**, 303], where it is not required that k be a finite field. (Naturally the author shows how the connection can be made between his theory and the slightly different theory of the field Ω_k.) For the treatment of the algebraic geometry on a curve the author permits himself to draw freely upon his book. Thus in the definition of the canonical divisors and in the proof of the theorem of Riemann-Roch the geometry of the 2-fold product of Γ by itself is used, and thereby tools like the intersection product, etc., are employed. Such tools are not absolutely necessary if one just desires to demonstrate the theorem of Riemann-Roch. However, for the discussion of

the all-important trace function it is preferable to have available, for example, the author's definition of a canonical divisor [theorem 8 on page 42]. The correspondences on Γ are introduced as divisors on $\Gamma\times\Gamma$, and the uniqueness of the product of correspondences is established [theorem 6 on page 35] leading to a simple formula for the latter. This work and the discussion of the additive group of correspondences with the concept of equivalence require a comprehensive part of the general theory of cycles and of the intersection product in chapters VII and VIII of the author's book.

The ring of correspondences $\mathfrak{A}=\{\xi, \cdots\}$ is introduced by means of the equivalence relation and a symmetry operator $\xi\to\xi'$ is defined on it essentially by the interchange of the factors in $\Gamma\times\Gamma$; the trace $\sigma(\xi)$ is defined by means of intersection multiplicities. The function σ has all the formal properties of a trace [theorem 7 on page 41]. The most complicated part of the paper consists in proving that $\sigma(\xi\xi')>0$ for nonzero ξ. Application of this basic inequality yields the positiveness of the quadratic form $\sigma(\xi\xi')=2gx^2+2\sigma(\iota^n)xy+2gq^ny^2$, where $\xi=x\delta+y\iota^n$ with integers x, y. In this formula g denotes the genus of Γ, δ denotes the identity correspondence and ι is the class of the correspondence $I(P)=P^\omega$, for each point P of Γ, where ω is the normalized automorphism of the algebraic completion $\bar{k}$ of k, $a^\omega=a^q$ for a in $\bar{k}$. The points P of Γ are essentially the same thing as the prime divisors of the coefficient extension $\Omega_k\bar{k}$. To each P there belongs, relative to k, a smallest field of definition $k(P)$. Then $d\log Z(u)=\sum_{n=1}^{\infty}\nu_n u^n du/u$, where ν_n is the number of distinct points P_j with $k(P_j)\subseteq k_n$, for the extension k_n of degree n over k. The numbers ν_n are identified with the numbers of components of certain intersection cycles related to $P\to P^{\omega^n}$. Thus the author shows $\nu_n=1+q^n-\sigma(\iota^n)$ and the expansion $d\log[(1-u)(1-qu)Z(u)]=\sum_{n=1}^{\infty}\sigma(\iota^n)u^n du/u$ together with the positiveness of $\sigma(\xi\xi')$ imply the Riemann hypothesis by means of a simple argument on analytic functions.

In the last paragraph further consequences of the structure of the ring $\mathfrak{A}$ are developed. Implications on the theory of L-series of function fields are given, and the connections with the groups of Hilbert and Artin's theory of the conductor are established. The higher ramification groups are given an interesting definition by means of the multiplicity of a point in a transformed cycle. These results depend on the identification of σ with the trace of a matrix representation of $\mathfrak{A}$ in a field of characteristic 0. The author announces the early publication of the pertinent facts which depend on the structure of the class group of Ω_k and further properties of the Jacobian variety attached to Γ, which incidentally already had to be used for the proof of the positiveness of $\sigma(\xi\xi')$ [pages 49–53]. *O. F. G. Schilling.*

d'Orgeval, B. Remarques sur la rationnalité des variétés. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **4**, 701–706 (1948).

If a rational three-dimensional variety V_3 is transformed birationally into the S_3, the following linear systems of surfaces will correspond to each other: (1) the anticanonical surfaces of V_3 which pass through the fundamental locus on V_3; (2) the quartic surfaces F_4 in S_3 which pass simply through the fundamental curves and doubly through the isolated fundamental points. Therefore, in order to prove that a given V_3 is not rational, it is sufficient to show that it is not possible to assign base (fundamental) loci on V_3 and in S_4 in such a fashion that the corresponding systems (1) and (2) are equivalent. On the other hand, if the V_3 is rational, then the consideration of the system (1), for a suitable choice of the base locus, may lead directly to a desired space representation of the V_3. This second possibility is illustrated by the example of the intersection of two quadrics in S_4. The author then proceeds to prove the irrationality of the general V_3^3 in S_4. He finds 17 possible base conditions for the system (2), and finds that each of these has to be rejected for one reason or another. The proof is only outlined and is admittedly "pas exhaustive."

O. Zariski (Cambridge, Mass.).

Maroni, Arturo. Sulle rigate astratte. Univ. Roma. Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) **7**, 236–242 (1948).

It is shown that an abstract ruled surface F [in the sense of Severi, same Rend. (5) **2**, 1–32 (1941); these Rev. **8**, 340] which is not an abstract cone can be put in a one-to-one birational correspondence without exception with a ruled surface provided there are no exceptional curves of the first kind on F. If such curves are present, then it is possible to establish a birational correspondence between F and a ruled surface which does not transform any exceptional curve of the second kind into a point. Let F_1 and F_2 be two given abstract ruled surfaces which are birationally equivalent to each other but which are not abstract cones. Then it is possible to establish between F_1 and F_2 a birational correspondence which does not involve more than $4p+2$ exceptional curves of the second kind transformed into points, where p is the common genus of F_1 and F_2.

C. Chevalley (Paris).

Zappa, Guido. Sui sistemi continui di curve sopra una rigata algebrica. Giorn. Mat. Battaglini (4) **77**, 172–183 (1947).

Après avoir répété, en en modifiant légèrement les démonstrations, certains résultats de A. Maroni sur les surfaces réglées algébriques [Ist. Lombardo Sci. Lett. Rend. (2) **36**, 586–600 (1903)], l'auteur considère sur une surface réglée algébrique, possédant une infinité de courbes directrices A de dégré zero, un système linéaire complet de courbes, système que l'on peut mettre toujours sous la forme $|kA+G_\lambda|$, où k est le nombre de points communs à une courbe du système et à une génératrice rectiligne de la surface, et G_λ est un groupe de λ génératrices. En faisant varier la série linéaire complète $|G_\lambda|$, on peut considérer la totalité continue complète de courbes $\{kA+G_\lambda\}$ et la totalité continue complète de systèmes linéaires $\{|kA+G_\lambda|\}$. L'auteur détermine tous les cas où un système $|kA+G_\lambda|$ est "exorbitant," c'est-à-dire les cas où il n'est pas contenu entièrement dans $\{kA+G_\lambda\}$; et les cas où il est "exubérant," c'est-à-dire les cas où il a une dimension plus grande que celle d'un système générique $|kA+G_\lambda|$, tout en étant contenu dans $\{kA+G_\lambda\}$. Les résultats obtenus permettent de trouver tous les systèmes continus $\{A+G_\lambda\}$ de courbes directices dont la série caractéristique, sur une courbe du système, est incomplète; et de discuter aussi le "défaut" de la série caractéristique du système linéaire $|A+G_\lambda|$ en cherchant les cas où ce défaut est plus petit que l'irrégularité de la surface. *E. G. Togliatti* (Gênes).

Wilson, E. M. Some enumerative polar properties of curves and surfaces. J. London Math. Soc. **22** (1947), 290–295 (1948).

The author exhibits a method of studying the polar k-points (for various values of k) of plane algebraic enve-

lopes, or of surface envelopes in space, in particular the common polar k-points of two or more envelopes, by using the well-known representation of the totality of plane envelopes Γ^n by the points of S_N, $N=\frac{1}{2}n(n+3)$, and known enumerative formulae. Besides obtaining the verifications of known results the author demonstrates two theorems. (I) Three general quartic envelopes have four common polar 9-points, i.e., their equations can be expressed linearly in terms of the same nine fourth powers in just four ways. (II) Two general quaternary cubics can be expressed linearly in terms of the same eight cubes of linear forms in just three ways. *M. Piazzolla-Beloch* (Ferrara).

Longo, C. Sui sistemi di ipersuperficie di S_r aventi lo stesso sistema primo polare. Univ. Roma. Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) **7**, 243–273 (1948).

Bertini [Atti Accad. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (5) 7_2, 217–227, 275–278 (1898)] étudiant ce problème montra que pour deux telles hypersurfaces V^n_{r-1} non coniques, il existait entre les pôles une homographie H, non dégénérée, que les V^n se distribuaient en systèmes linéaires et il donna un mode de construction de ces systèmes à partir de formes possédant pour espaces n-ples les conjugués des espaces unis de H. Il donna aussi un critère analytique: si f et g sont deux hypersurfaces d'un système et si H est $y_i=\sum a_{ik}x_k$ on a $\sum a_{ik}dg/dx_idx_j=\sum a_{ik}df/dx_idx_i$. Il détermina de plus les systèmes où $r=1, 2, 3$ et caractérisa géométriquement les deux premiers. L'auteur étend ces résultats à des homographies particulières. Au cas d'une homographie avec une seule chaîne, $y_i=a_ix_{i-1}+ax_i$, il résoud les équations de Bertini, construisant une solution, somme de la solution du problème à $r-1$ dimensions et d'une fonction Γ calculable par récurrence; ceci conduit à un seul système linéaire. L'étude de Γ montre que les V passent par espaces unis S_{r-h} avec la multiplicité $r+1-[r/h+1]$ et ont au point uni un point $(r-1)$-fois uniplanaire avec le S_{r-1} tangent $r-1$ fois; il détermine ainsi une calotte dont il calcule les éléments de contact au sens de Bompiani [Univ. Roma. Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) **5**, 1–46 (1946); ces Rev. **9**, 60]. Étudiant les cas simples $r=1, 2, 3, 4$ il montre que la caractérisation du cas général peut se faire à l'aide des polaires successives des points unis et des invariants de contact des calottes définies plus haut. Dans le cas d'une homographie avec h chaînes de q éléments il construit une solution somme d'une forme par rapport aux premières variables de chaque chaîne et d'une fonction calculable par récurrence. Les cas les plus simples laissent prévoir que la caractérisation dépend des mêmes éléments que précédemment et de certaines quadriques contenant les espaces unis, ceux des variables d'une chaîne et les espaces en joignant les points. Quelques indications sur le cas général permettraient de le ramener à des combinaisons linéaires de solutions de chaque chaîne. *B. d'Orgeval.*

Godeaux, Lucien. Sur une surface de rang p n'ayant que des points de diramation doubles biplanaires. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **34**, 115–121 (1948).

In earlier papers the author has studied cyclic involutions associated with algebraic surfaces. In the present paper he treats a special case of such an involution of order p in which the image surface Φ of order $2p$ has $2p^2$ branch-points and no other singularities. This case is of interest chiefly because, in its development, the equation is obtained of a family of surfaces which have contact of order $p-1$ with Φ along a known curve. *T. R. Hollcroft.*

Godeaux, Lucien. Structure des points unis des involutions cycliques appartenant à une surface algébrique. C. R. Acad. Sci. Paris **227**, 173–175 (1948).

L'auteur considère une surface algébrique F, contenant une involution cyclique I_p d'ordre premier p, et comme modèle projectif de F une surface normale d'ordre pn de S_r sur laquelle I_p est déterminée par une homographie cyclique H possedant p axes pontuels $\sigma_0, \sigma_1, \cdots, \sigma_{p-1}$. Il désigne par $|C_i|$ le système de courbes découpé sur F par les hyperplans de S_r passant par $\sigma_0, \sigma_1, \cdots, \sigma_{i-1}, \sigma_{i+1}, \cdots, \sigma_{p-1}$. Soit A un point uni. En considérant des courbes C_i particulières passant par A, il montre que la détermination de la structure du point uni A_N équivaut à déterminer le comportement de ces courbes en A. *M. Piazzolla-Beloch.*

Godeaux, Lucien. Sur une surface représentant une involution cyclique du quatrième ordre. Math. Notae **7**, 205–211 (1947).

L'auteur donne un exemple d'involution cyclique du quatrième ordre, appartenant à une surface F de l'espace à cinq dimensions, d'équations

$$x_4^4=\varphi_1(x_0,x_1,x_2,x_3),\quad x_5^4=\varphi_2(x_0,x_1,x_2,x_3),\quad x_4x_5=\varphi(x_0,x_1,x_2,x_3),$$

où φ_1, φ_2 sont des formes du quatrième degré, et φ une forme algébrique du second degré. La surface F est transformée en elle-même par l'homographie cyclique de période quatre, d'équations

$$x_0':x_1':x_2':x_3':x_4':x_5'=x_0:x_1:x_2:x_3:ix_4:-ix_5.$$

Cet homographie engendre sur F une involution I_4 d'ordre quatre. La surface F' image de cet involution, du huitième ordre, appartient à l'espace ordinaire et possède, en chaque point de diramation, un point double biplanaire, auquel est infiniment voisin un point double conique. L'auteur donne ensuite l'équation de certaines familles de surfaces ayant un contact du troisième ordre avec la surface image le long de certaines courbes. *M. Piazzolla-Beloch* (Ferrara).

Piazzolla-Beloch, M. Sulle proprietà topologiche dei circuiti d'ordine dispari tracciati sopra quadriche a punti iperbolici. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **4**, 688–691 (1948).

Differential Geometry

***Federer, Herbert. An Introduction to Differential Geometry.** Distributed by the Stenographic Bureau, Brown University, Providence 12, R. I., 1948. Unpaged. $5.00.

These are mimeographed notes [about 450 pp.] of a course at Brown University. The subject matter is not that usually implied by the title. It is not concerned with metric differential geometry but rather with the more primitive notions of differential forms, their integrals, de Rham's theorem and related matters. The author has performed a sizable task by presenting, for the first time, an organized account of material so widely distributed in the literature.

The book divides into four chapters. The first treats the classical material of finite dimensional vector spaces, quadratic forms, characteristic polynomials and normal forms of linear transformations, and the theory of Grassmann algebras.

The second chapter presents the definitions and elementary properties of differentiable manifolds, differentiable

maps, regular and proper maps, and differential forms on manifolds. The main theorem is that of Whitney [Ann. of Math. (2) **37**, 645–680 (1936)]: a differentiable n-manifold may be differentiably imbedded in Euclidean $(2n+1)$-space. The proof has several improvements over that of Whitney; but is restricted to manifolds of class not less than 2, and it is not shown that the image manifold admits an analytic structure.

The main result of chapter 3 is the theorem of de Rham [J. Math. Pures Appl. (9) **10**, 115–200 (1931)] that the factor group of the exact k-forms by the derived k-forms is isomorphic to the k-dimensional cohomology group of the manifold. The proof avoids triangulating the manifold in the following manner. The manifold M is imbedded in a Euclidean space E and a neighborhood U of M in E is selected which retracts into M under normal projection. Groups of U are defined isomorphic, under the projection, to the groups of M in question. This reduces the proposition in M to one in U, and U is obviously triangulable.

The last chapter is entitled Differential Equations, and is concerned with the infinitesimal transformations associated with vector fields, Pfaffians, differential ideals, their characteristic vectors, first integrals, and integral structures.

A notable feature of the book is the use of invariant methods throughout. An extreme example of this is the definition of the determinant of a linear transformation without the use of a base. The definition of a differentiable manifold follows the method of Chevalley [Theory of Lie Groups I, Princeton University Press, 1946; these Rev. **7**, 412] where both the topology and differential structure are carried by a family of functions defined on subsets of a set M. Thus a (contravariant) vector at a point p of M is defined to be a linear function of functions defined near p satisfying the product rule for differentiation. A skew k-form (alternating covariant tensor of order k) is a function of ordered k-tuples of vectors.

The most striking feature of the book to the casual reader is the notation. The author adopts the view that certain familiar notations are misleading, and obscure the meanings of definitions and theorems. He replaces them by more elaborate notations based on the roots in set theory of the concepts represented (e.g., the polynomial x becomes the sequence of its coefficients $[0, 1]$). A few such changes would not be worthy of comment; but he has carried out the prodigious task of applying the same stern standards to every phase of the work. The result can be described by saying that a resemblance to any notation, living or dead, is purely coincidental.

There is no index, and one is sorely needed. A serious reader would do well to make one as he goes along.

N. E. Steenrod (Princeton, N. J.).

Zwirner, G. Alcuni teoremi di geometria infinitesimale diretta relativi alle curve spaziali. I. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **4**, 524–530 (1948).

If a curve C in Euclidean 3-space is the homeomorph of a line segment, and has at each point a tangent parallel to a fixed plane p (alternative: passing through a fixed line L), it is proved that C is in a plane parallel to p (alternative case: in a plane passing through L). Two similar theorems, with essentially stronger hypotheses, are also given. Taking p as the (x, y)-plane, a main step in the proof is to show that any given arc of the curve contains a subarc on which z is a single-valued function of x, or a single-valued function of y. The following theorem of Baire is used in this step: if R is a complete metric space, then the intersection of countably many open sets which are dense in R, is dense in R. [Cf. Alexandroff and Hopf, Topologie I, Springer, Berlin, 1935, p. 108.] *A. B. Brown* (Flushing, N. Y.).

Sz. Nagy, Gyula. Über die Lage der Doppelgeraden von gewissen Flächen gegebener geometrischer Ordnung. Acta Univ. Szeged. Sect. Sci. Math. **11**, 234–238 (1948).

The author studies closed real surfaces F which satisfy certain regularity conditions. In addition, they are assumed to contain neither planes nor ruled quadrics. The (geometric) order of F is the maximum number of its points of intersection with a straight line that does not lie in F completely. These intersections are provided with multiplicities so that multiple straight lines, etc., can be defined on F. The author proves and generalizes various observations on the position and number of double straight lines on an F of order four or five. The proofs follow immediately from his assumptions on F. *P. Scherk* (Saskatoon, Sask.).

Havlíček, Karel. Contact des courbes et des hypersphères dans un espace euclidien à n dimensions—courbes sphériques. Časopis Pěst. Mat. Fys. **72**, 137–146 (1947). (French. Czech summary)

Notations: R_q is a q-dimensional linear space; V_1 is a curve in R_n with the parametric representation $\mathbf{r}=\mathbf{r}(s)$; $\rho_1, \cdots, \rho_{n-1}$ are the radii of curvature of V_1; $\Sigma_\nu(\Sigma_\nu^0)$ is a ν-dimensional hypersphere (having a contact of order q with V_1); Σ_q^{q+1} is a q-dimensional osculating hypersphere; $\mathbf{n}_1, \mathbf{n}_2, \cdots, \mathbf{n}_{n-1}$ are mono-, bi-, tri- $\cdots$ normal unit-vectors at a point of V_1; only regular points of V_1 are considered such that $(dr/ds)^2 \neq 0$. The author proves that the centers of all the Σ_{q-1}^q lie in an R_{n-q} having the equation $\mathbf{r}=\sum_{\lambda=1}^{q-1} c_\lambda{}^0 \mathbf{n}_\lambda + \sum_{\nu=q}^{n-1} c_\nu \mathbf{n}_\nu$, where the c_ν are independent parameters, and the constants $c_\lambda{}^0$ (the coordinates of the center of Σ_{q-1}^q) are determined as functions of the $d^\lambda \rho_\mu / ds^\lambda$ by linear equations. Hence the radius of Σ_{q-1}^q is given by $R_q^2=\sum_{\nu=1}^{q-1}(c_\nu{}^0)^2$. A necessary and sufficient condition that V_1 should lie on a Σ_{n-1} is that $\rho_1, \cdots, \rho_{n-1}$ satisfy the differential equation $a_n - a_{n-2}=0$, where the a_i are defined by the recurrent relations $a_0=0$; $a_1=\rho_1$; $a_i=\rho_i da_{i-1}/ds + a_{i-2}\rho_i/\rho_{i-1}$ $(i=2, 3, \cdots, n-1)$, $a_n=-\rho_{n-1} da_{n-1}/ds$. A V_1 for which $\rho_i=\text{constant} \neq 0$ $(i=1, 2, \cdots, n-1)$ is a hypercircle, i.e., it lies on a Σ_{n-1}, if and only if n is even. *E. Egerváry.*

Efimov, N. V. Research on the deformation of surfaces containing points with zero Gaussian curvature. Mat. Sbornik N.S. **23**(65), 89–125 (1948). (Russian)

For fixed t, $0 \leqq t \leqq 1$, let $S_t: x_i=x_i(u, v, t)$, $u^2+v^2<1$, be a surface in E^3 for which the second partial derivatives with respect to u and v are continuous and the Jacobians $\partial(x_i, x_j)/\partial(u, v)$ vanish nowhere simultaneously. If $x_i(u, v, t)$, $x_{i\,v}(u, v, t)$, $x_{i\,v}(u, v, t)$ are uniformly continuous in t then the family S_t yields a uniform deformation of S_0 into S_1. If S_t has negative curvature for $0<u^2+v^2<1$, then the mapping of S_t on the unit sphere by the spherical image is locally topological for $0<u^2+v^2<1$. Its index is independent of t [under the assumption that the second derivatives of the x_i depend continuously on t the theorem was proved by H. Schilt, Compositio Math. **5**, 239–283 (1937)]. There are intrinsically isometric surfaces S, S' with corresponding points p, p' such that no neighborhood of p on S can be uniformly deformed into the corresponding neighborhood of p' on S'.

Let $S: X(u, v)$ be an analytic surface with line element ds^2 and Gauss curvature $K(u, v)$. A set $\{U\}$ of analytic sur-

faces U in E^3 defined in a neighborhood $u^2+v^2<a^2$, $a>0$, and with the same line element ds^2 as S is called connected if it contains with any surface its images under reflections in planes and if for U_1, $U_2 \varepsilon \{U\}$ the surface U_1 can be bent into U_2 or an image of U_2 under reflection within $\{U\}$. If $K(0,0)\neq 0$ then E. Levi [Atti Accad. Sci. Torino **43**, 292–302 (1908)] showed that the set $\{U_0\}$ of all analytic surfaces with the ds^2 as line element is connected for a suitable $a>0$. The statement is no longer true if $K(0,0)=0$, but it remains true for the subset $\{U_1\}$ of $\{U_0\}$ consisting of those surfaces which have first order contact with their tangent planes at $(0,0)$ [Schilt, loc. cit.]. Let $\{U_2\}$ be the subset of $\{U_1\}$ for which a neighborhood of $(0,0)$ exists such that the spherical image of $(0,0)$ does not coincide with the spherical image of another point. The subset $\{U_2\}$ is connected if and only if S possesses at most one geodesic through O along which $K_u=0$ and $K_v=0$.

H. Busemann (Los Angeles, Calif.).

Löbell, Frank. Aus der Theorie der Flächenabbildungen. Arch. Math. **1**, 73–76 (1948).

This is a report on a series of publications dealing with mappings of two surfaces on one another [cf. S.-B. Math.-Nat. Abt. Bayer. Akad. Wiss. **1944**, 107–132; **1945/46**, 175–183 (1947); these Rev. **9**, 614]. *P. Scherk.*

Salini, Ugo. Sulle trasformazioni puntuali fra due spazi ordinari in una coppia ad jacobiano nullo di caratteristica uno. Univ. Roma. Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) **7**, 44–71 (1948).

Study of a correspondence between two projective spaces in the neighborhood of a pair of corresponding points O, O' where the Jacobian vanishes to the first order. A projective reference between the bundles of lines through O, O' is determined by their corresponding neighborhoods of the second order. Some invariants and invariant geometrical configurations (lines, planes, involutions) are given.

E. Bompiani (Pittsburgh, Pa.).

Su, Buchin. On certain cones connected with a nonholonomic surface in affine space. Tôhoku Math. J. **48**, 225–234 (1941).

This is a generalization to a nonholonomic V_2^3 in affine space of previous results of the author on a V_2^3 in ordinary space and of the author and T. Kubota on holonomic surfaces. The affine normals at O to the sections of V_2^3 by a plane through a nonasymptotic tangent at O lie in a plane, the generalization of the plane of Transon. When the tangent t at O varies in the tangent plane the corresponding plane of Transon envelopes an algebraic cone of order 4 and class 3 with three cuspidal generators. The three cuspidal tangent planes are concurrent in a line N, the affine normal of V_2^3 at O. These three planes meet the tangent plane of V_2^3 in the Segre tangents. Furthermore, by use of Moutard quadrics and Transon planes a one-parameter family of fourth order cones $\Gamma_4(k)$ are covariantly definable leading by the same construction as above to a family of lines $N(k)$ lying on a quadratic cone. Each $N(k)$ is the polar of the tangent plane with respect to the trihedron formed by the three cuspidal generators of $\Gamma_4(k)$. *J. L. Vanderslice.*

Kovancov, N. I. A generalization of certain constructions of projective differential geometry. Doklady Akad. Nauk SSSR (N.S.) **61**, 973–976 (1948). (Russian)

In the projective differential geometry of surfaces there are many constructions that lead to certain canonical lines such as the normal of Fubini, Green's edge, etc. The author gives a generalization of their constructions which lead to pencils of canonical lines. In such cases the line is based on a metric form, the normal of Fubini depending on the form $2\beta\gamma du dv$, the form $\beta\gamma(\beta du^2+\gamma dv^2)$ gives the directrix of Wilczynski, while the ratio of the two gives the Cartan line. By considering the general form $f(\beta,\gamma,du,dv)$ the author finds that, in order that it should yield a line, $f=\beta^{\alpha}\gamma^{t}du^{2\alpha-t}dv^{2t-\alpha}+\beta^{t}\gamma^{\alpha}du^{2t-\alpha}dv^{2\alpha-t}$, where t and α are arbitrary numbers, thus generalizing Fubini's construction. In a similar manner the author generalizes the constructions of Green and Čech and finds that the classes of lines obtained by these two constructions coincide. *M. S. Knebelman.*

Ruse, H. S. On simply harmonic "kappa-spaces" of four dimensions. Proc. London Math. Soc. (2) **50**, 317–329 (1948).

Un espace riemannien est un "kappa-espace" K_n s'il existe un champ de vecteurs κ_p tel que $\nabla_p R_{ijkl}=\kappa_p R_{ijkl}$ où ∇_p est l'opérateur de dérivation covariante. Ce sont donc des généralisations des espaces riemanniens symétriques de É. Cartan. Ils ont été introduits par l'auteur dans un travail antérieur [J. London Math. Soc. **21** (1946), 243–247 (1947); ces Rev. **9**, 102] où figure aussi une condition nécessaire et suffisante simple pour qu'un K_n soit simplement harmonique, c'est-à-dire pour que l'équation de Laplace associée admette la solution élémentaire s^{2-n}. Le présent papier est consacré d'une part à l'étude des K_4 généraux, d'autre part à la détermination complète de ceux d'entre eux qui sont simplement harmoniques. Il est montré que tout K_4 admet un tenseur de courbure de la forme $R_{ijkl}=l_{ij}l_{kl}+m_{ij}m_{kl}+n_{ij}n_{kl}$ où l_{ij}, m_{ij}, n_{ij} sont trois bivecteurs simples satisfaisant aux conditions $\epsilon^{ijkl}l_{ij}m_{kl}=0$ etc. et $\epsilon^{ijkl}l_{ij}\kappa_k=0$ etc., dont l'auteur donne une interprétation géométrique à l'aide du complexe de Riemann. Dans les cas de dégénérescence de la conique de base du complexe on peut avoir $n_{ij}=0$ ou $n_{ij}=l_{ij}=0$. Ce dernier cas est celui qui se produit nécessairement pour les K_4 simplement harmoniques. On en déduit que pour de tels K_4, le vecteur κ_p est le gradient d'un scalaire κ satisfaisant à $\Delta_1\kappa=0$, $\Delta_2\kappa=0$. Par une étude des champs de vecteurs parallèles, l'auteur montre que les K_4 simplement harmoniques sont ceux pour lesquels il existe un système de coordonnées ramenant la métrique à la forme

$$ds^2=\alpha dx^2+2\beta dxdy+\gamma dy^2+2dxdz+2dydt$$

où α, β, γ sont des fonctions arbitraires de x et y. On notera que ce K_4 a la signature $++--$; car, d'après des résultats de A. G. Walker et du reviewer, les espaces V_n simplement harmoniques à métrique définie ou hyperbolique normale se réduisent nécessairement à l'espace euclidien.

A. Lichnerowicz (Strasbourg).

Allendoerfer, C. B. Global theorems in Riemannian geometry. Bull. Amer. Math. Soc. **54**, 249–259 (1948).

Vortrag vor der American Mathematical Society. Nach einem kurzen Bericht über bekannte Hauptprobleme aus der Differentialgeometrie im Grossen wird der von Allendoerfer-Weil, Fenchel und Chern bewiesene Satz besprochen, dass die Eulersche Zahl einer geschlossenen Riemannschen Mannigfaltigkeit gerader Dimension durch das Integral über einen Krümmungsskalar ausgedrückt werden kann. Es wird besonders Cherns Methode hervorgehoben, welche mit Vektorfeldern arbeitet, und es wird betont, dass die geometrische Bedeutung des Randintegrals in der Gauss-Bonnet'schen Formel für $2n$ Dimensionen noch nicht geklärt ist (für $n=1$ ist es das Integral der geodätischen Krüm-

mung). Die wichtige Aufgabe, auch Cohn-Vossens Satz über die Curvatura Integra offener Flächen auf mehr-dimensionale Mannigfaltigkeiten zu übertragen, wird formuliert. Mehrere Möglichkeiten, Krümmungsskalare zu definieren, werden erwähnt. Am Schluss wird kurz auf Hodges Theorie der harmonischen Integrale hingewiesen.

Berichtigungen: (1) zu § 1, Zeile 18: Lausanne liegt in der Schweiz; (2) zu p. 250, Zeile 13: die lokale Kompaktheit ist trivial; gemeint ist, dass in einer vollständigen Mannigfaltigkeit M jede Menge, die im Sinne der Metrik beschränkt ist, kompakt rel. M ist; (3) zu p. 251, Zeile 15: Weyls zitierte Arbeit behandelt die infinitesimale Starrheit, aber nicht die Einzigkeit der geschlossenen konvexen Flächen; der Einzigkeits-Satz ist von Cohn-Vossen [Nachr. Ges. Wiss. Göttingen. Math.-Phys. Kl. 1927, 125–134] bewiesen worden. *H. Hopf* (Zürich).

Michal, Aristotle D. Global groups of motions of some infinitely dimensional Riemannian spaces. Bull. Soc. Math. Grèce 23, 143–151 (1948).

Consider the space C of continuous functions of two variables defined on the square $a \leqq x \leqq b$, $a \leqq y \leqq b$, and let Δ be the (open) set of such "kernels" with non-vanishing Fredholm determinants. For each $\beta \epsilon \Delta$ let β' denote the resolvent kernel of β. Juxtaposition of kernels will denote Volterra's integral composition, thus $\alpha\beta = \gamma(x, y) = \int_a^b \alpha(x, s)\beta(s, y)ds$. For each $\alpha, \beta \epsilon C$ define an inner product $[\alpha, \beta] = \int_a^b dx \int_a^b \alpha(s, x)\beta(s, x)ds$ which is obviously positive definite. Finally, put $ds^2 = [\Omega(\beta, \delta\beta), \Omega(\beta, \delta\beta)]$, where $\Omega(\beta, \delta\beta) = -\delta\beta - (\delta\beta)\beta'$. Then ds can be considered as the element of arc length for an infinitely dimensional Riemannian space with "coordinates" in the open set Δ of the function space C. [For a discussion of generalized Riemannian spaces see Michal, Bull. Amer. Math. Soc. 45, 529–563 (1939); these Rev. 1, 29.] It is proved that the group of functional transformations $\bar{\beta} = \beta + \alpha + \beta\alpha$ is a group of motions for this infinitely dimensional Riemannian space, and similarly that $\bar{\beta} = \beta + \alpha + \alpha\beta$ defines a group of motions for the space based on the differential form obtained by replacing Ω by $\Phi(\beta, \delta\beta) = \delta\beta + \beta'\delta\beta$. Three groups of motions are obtained for the "symmetric" space (a generalization of É. Cartan's symmetric space) which is shown to be a generalized Riemannian space with an indefinite metric form. The proof depends on the solution of a generalized Pfaffian equation. *D. H. Hyers* (Los Angeles, Calif.).

Gołąb, St. Sur la théorie des objets géométriques. (Réduction des objets géométriques spéciaux de première classe aux objets du type D). Ann. Soc. Polon. Math. 20 (1947), 10–27 (1948).

The author considers geometric objects with one component u of the first class, which means geometric objects transforming under coordinate transformations $\xi^{\lambda'} = \phi^{\lambda'}(\xi^i)$; $h, i = 1, \cdots, n$, as follows: $u' = f(x, A_i^{\lambda'})$; $A_i^{\lambda'} = \partial_i \xi^{\lambda'}$. He arrives at the result that for $n \geqq 2$ these objects are of type Δ, which means that $f(x, A_i^{\lambda'}) = \phi(x, \Delta)$, $\Delta = \det(A_i^{\lambda'})$. This result, however, is not correct for $n = 2$. The object formed by the ratio of the components of a vector gives a counter-example. It seems that the author has made a mistake in computing the Poisson brackets of the system (96) for $n = 2$. The sixth equation of (98) should read $(X_2X_4)f = X_3 f$. The same problem has been treated in a somewhat more general way by G. Pensov [C. R. (Doklady) Acad. Sci. URSS (N.S.) 54, 563–566 (1946); these Rev. 9, 67] using a method introduced by V. Wagner [ibid. 46, 347–349 (1945); these Rev. 7, 265]. *J. Haantjes.*

Gheorghiu, Octavian Emil. Équations aux dérivés partielles et objets géométriques. C. R. Acad. Sci. Paris 227, 613–615 (1948).

L'auteur met en évidence les objets géométriques associés à une équation aux dérivées partielles linéaire du second ordre $\theta(f) \equiv b^{ij}\partial_i\partial_j f + c^i\partial_i f + af = 0$ dans l'hypothèse que f et $\theta(f)$ se transforment sous une transformation

$$x^{i'} = x^{i'}(x^1, \cdots, x^n); \quad \Delta = \det(\partial x^{\lambda'}/\partial x^i),$$

selon $\bar{f} = f\Delta^p$; $\bar{\theta}(\bar{f}) = \theta(f)\Delta^p$. *J. Haantjes* (Leiden).

Funk, Paul. Beiträge zur zweidimensionalen Finsler'schen Geometrie. Monatsh. Math. 52, 194–216 (1948).

L'auteur recherche à quelles conditions un problème de calcul des variations à deux dimensions, défini par $\int f(x, y, y')dx$, peut être considéré comme déduit, par une transformation ponctuelle du plan, d'un problème analogue dont les extrémales soient des droites. Le même problème a été résolu au moyen d'un méthode tensorielle par Berwald [Ann. of Math. (2) 42, 84–112 (1941); ces Rev. 2, 304]. Le présent travail, dont l'édition s'est trouvé retardée, est contemporain du travail de Berwald mais utilise une méthode bien différente, basée essentiellement sur l'usage d'un système de coordonnées particulier lié au problème. Par un élément linéaire L_0, de centre P_0, on mène l'extrémale C_0. A partir de C_0, on définit la fonction "latitude" relative à C_0, $b(x, y)$, nulle sur C_0 et satisfaisant à l'équation $b_y' = [f(x, y, -b_x'/b_y') \cdot f_{y'y'}(x, y, -b_x'/b_y')]^{\frac{1}{3}}$ qui s'introduit aisément à l'aide de la condition de transversalité. Désignons par C les courbes b = constante. Si P est un point voisin de C_0, ses coordonnées seront le b correspondant à la courbe C passant par P et l'abcisse curviligne S sur C_0 du point Q où la transversale à C issue de P coupe C_0. A l'aide de ce système de coordonnées, le système des deux conditions nécessaires et suffisantes cherchées est obtenu d'une manière relativement aisée. Chemin faisant, quelques formules de géométrie finslerienne à deux dimensions sont trouvées.

A. Lichnerowicz (Strasbourg).

NUMERICAL AND GRAPHICAL METHODS

***Rosser, J. Barkley. Theory and Application of $\int_0^z e^{-x^3}dx$ and $\int_0^z e^{-x^3}dx \int_0^z e^{-x^3}dx$. Part I. Methods of Computation.** Mapleton House, Brooklyn, N. Y., 1948. iv+192 pp. $8.00.

This gives a thorough theoretical account of the functions named in the title, z and p taking general complex values. The formulae include various expansions in powers of z and asymptotic expansions in powers of $1/z$. There are also sections on continued fraction expansions and the connection with Hermite polynomials, etc. These were all developed with a view to computing the functions, and coefficients (rational) in most series are given to 10 or more terms.

The very useful tables are confined to the cases where the phase of z is a multiple of $\frac{1}{3}\pi$, and where $p^3 = -1$, i.e., they deal with the error integral, Dawson's integral, and the Fresnel integrals and a certain further integral involving each of these functions.

Table 1 gives $Rr(u)$, $Ri(u)$,

$$[Rr(u)]^2+[Ri(u)]^2=\tfrac{1}{2}-2\int_0^u Ri(x)dx,$$

$\int_0^u Rr(x)dx$, where

$$Rr(u)+iRi(u)=(2/\pi)^{\frac{1}{2}}\int_0^\infty e^{-\frac{1}{2}\pi ux^2}(1-ix^2)^{-1}dx$$

$$=ie^{\frac{1}{2}i\pi u^2}[\tfrac{1}{2}-C(u)-i\{\tfrac{1}{2}-S(u)\}],$$

in which $C(u)$ and $S(u)$ are Fresnel integrals given by

$$C(u)+iS(u)=\int_0^u e^{\frac{1}{2}i\pi t^2}dt.$$

Values are given to 12 decimals for $u=-.06(.02)3.5(.05)5.15$, except for $\int_0^u Rr(x)dx$, which is given to $u=5$ only.

Table 2 gives $e^{-w^2}\int_0^w e^{v^2}dv$, $e^{w^2}\int_w^\infty e^{-v^2}dv$, and the integrals from 0 to w of these functions, to 10 decimals for $w=-.2(.05)3.5(.1)6$, with a few extra values of the first two functions (a) between 3.5 and 4, and (b) from 6 to 6.3; the first integral is carried on for $w=6(.1)6.5(.5)12.5$.

Interpolation is discussed, and diagrams indicate the range of validity of various Lagrangean formulae.

The production of the tables (by photographing from typescript) is so poor that a special comment on the lack of care is deserved. On pp. 187 and 189, the left and right halves of the page should be interchanged.

J. C. P. Miller (London).

Goodwin, E. T., and Staton, J. Table of $\int_0^\infty \frac{e^{-u^2}}{u+x}du$. Quart. J. Mech. Appl. Math. 1, 319–326 (1948).

The function $f(x)$ of the title is tabulated to four decimals for the range $x=0(0.02)2(0.05)3(0.1)10$. To facilitate interpolation for small x an auxiliary function $g(x)=f(x)+\log x$ is tabulated to four decimals for $x=0(0.01)1$. An asymptotic expression is given for values of x greater than 10. Full details of the method of computation are given. Repeated Euler transformation applied on the asymptotic series gives the value of $f(1)$ to seven decimals.

J. G. van der Corput (Amsterdam).

Duarte, F. J. On Everett's interpolation formula. Estados Unidos de Venezuela. Bol. Acad. Ci. Fís. Mat. Nat. 10, 379–385 (1947). (Spanish)

This note gives a well-known derivation of Everett's interpolation formula. *T. N. E. Greville.*

Serebrennikov, M. G. A more exact method of harmonic analysis of empirical periodic curves. Akad. Nauk SSSR. Prikl. Mat. Meh. 12, 227–232 (1948). (Russian)

It is assumed that $f(x)$, the curve of period 2π which is to be analysed, has no jumps or abrupt changes in direction. As usual, the points (x_j, y_j), $j=0, \cdots, 2n$, $x_{j+1}-x_j=\alpha=\pi/n$, on it are supposed to be known. An expression is obtained for the Fourier cosine coefficient A_{mj} of arbitrary order m, for the function $\varphi_j(x)$ which consists of the segment of the straight line joining (x_j, y_j) to (x_{j+1}, y_{j+1}) and coincides with the x-axis elsewhere from 0 to 2π. (For brevity the sine terms will not be mentioned.) By hypothesis the function $\eta(x)=f(x)-\varphi(x)$, where $\varphi(x)=\sum_{j=0}^{2n-1}\varphi_j(x)$, consists of $2n$ smooth arcs. Each of the individual arcs $\eta_j(x)$ of $\eta(x)$ is replaced by a cubic polynomial $\psi_j(x)$ having the same ordinate and slope at its ends as $\eta_j(x)$. The method of A. Eagle [A Practical Treatise on Fourier's Theorem and Harmonic Analysis, Longmans, Green, London, 1925, pp. 57–61, 102–111] is employed to write the exact value of the Fourier cosine coefficient a_m of $\psi(x)=\sum_{j=0}^{2n-1}\psi_j(x)$ in terms of $\delta_j^{(i)}$, $i=1, 2, 3$, the change in the ith derivative of $\psi(x)$ at the point x_j. The $\delta_j^{(i)}$ are in turn expressed in terms of the quantities $\tan\beta_j=(y_{j+1}-y_j)/\alpha$ and $\tan\epsilon_j=f'(x_j)$.

No limit is given for the error in taking $A_m^*+a_m$, where $A_m^*=\sum_{j=0}^{2n-1}A_{mj}$, as the final approximation to A_m, the desired Fourier cosine coefficient of $f(x)$, nor are the results of numerical examples shown for comparison with other methods. Suggestions for numerical calculation are not given. The author shows that the formulas can be extended to the case of unequally spaced ordinates with very little change and not much additional complexity.

An estimate of the labor involved in applying the method in the case of equally spaced ordinates is had by observing that A_m^* could be obtained from one ordinary harmonic analysis using the quantities $\Delta^2 y_{j-1}$ as the data [the author does not seem to notice that his formula (1.1) for A_m^* reduces to the simple result of Eagle on p. 103] while a_m would require three more such analyses with combinations of $\tan\beta_j$ and $\tan\epsilon_j$ as data. The reviewer is not convinced that in this case the proposed method gives more accurate results, for comparable effort, than would be obtained by measuring more ordinates (since presumably the curve would be available for measurement of the slopes of tangents, this would be possible) and using the ordinary methods of analysis. *R. Church* (Annapolis, Md.).

Wassermann, G. D. On perturbation problems associated with finite boundaries. Proc. Cambridge Philos. Soc. 44, 251–262 (1948).

The author treats the problem of perturbation of boundary conditions by means of the displacement operator technique which was first introduced by Brillouin [C. R. Acad. Sci. Paris 204, 1863–1865 (1937)] and Cabrera [C. R. Acad. Sci. Paris 207, 1175–1177 (1938)]. The main difference from previous treatments of this subject lies in the approximate solution of the differential equations satisfied by the corrections to the wave function by means of a variational procedure. An example in which the eigenvalues are found is the scalar Helmholtz equation in a region bounded by a regular hexagon with the boundary condition $\partial\varphi/\partial n=0$. Previous treatments are criticized. Unfortunately, these remarks seem to the reviewer to be at best only superficially cogent.

H. Feshbach (Cambridge, Mass.).

Morgantini, Edmondo. Teoria dei nomogrammi a punti allineati con due scale rettilinee o sovrapposte ad una stessa conica. Univ. Roma. Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) 7, 194–235 (1948).

Let F_{1234} be a given function of four variables z_1, z_2, z_3, z_4, the subscripts on any letter denoting the variables upon which the corresponding function depends. The paper under review concerns principally the nomogram got by writing $K_1K_2K_{34}F_{1234}$ equal to a determinant of order three whose three rows involve only the variable z_1, the variable z_2 and the variables z_3, z_4, respectively. The author gives in geometrical terms necessary and sufficient conditions for representation when the first two rows correspond to straight supports (type I) or to coincident conics (type II). The functions employed are differentiable. There is included some preliminary material on linear dependence of functions: to the references cited by the author in this connection

might be added Kellogg [Z. Math. Physik 63, 159–173 (1915)]. *J. M. Thomas* (Durham, N. C.).

Hartree [Hartree], D. R. The Eniac, an electronic computing machine. Uspehi Matem. Nauk (N.S.) 3, no. 5(27), 146–158 (1948). (Russian)

Translated from Nature 158, 500–506 (1946); these Rev. 8, 355.

Yanžul, I. N. Calculating automata and their application to astronomical computation. Uspehi Matem. Nauk (N.S.) 1, no. 5–6(15–16), 27–40 (1946). (Russian)

Akušskiĭ, I. Ya. Calculating-analytical machines and their application to mathematical problems. Uspehi Matem. Nauk (N.S.) 3, no. 5(27), 187–189 (1948). (Russian)

Summary of the author's dissertation.

MECHANICS

Zel'manov, A. L. The application of deforming coordinates in nonrelativistic mechanics. Doklady Akad. Nauk SSSR (N.S.) 61, 993–996 (1948). (Russian)

The Lagrangian particle labels x^1, x^2, x^3 can be used as time-dependent curvilinear space coordinates, and the $\partial/\partial x$ derivatives can be introduced into the equations of motion in the place of the Eulerian spatial ones. If $ds^2=h_{ik}x^ix^k$, and $R_{ik}=\partial u_i/\partial x_k-\partial u_k/\partial x_i$, the equation of continuity is $\partial(\rho h^1)/\partial t=0$, and the equation of motion $\partial R_{ik}/\partial t=\partial S_i/\partial x_k-\partial S_k/\partial x_i$, where S_i is the resultant vector of body forces and stresses per unit mass. If the space is Euclidean, the covariant curl, G_{ijk}, of $R_{ik}+\partial h_{ik}/\partial t$ vanishes. Compatibility conditions of the equations of motion are given in terms of the tensors h, G and the curvature tensor. Applications to astronomy are hinted at.

A. W. Wundheiler (Chicago, Ill.).

Kucharski, W. Zur Kinetik dehnungsloser Seile mit Knickstellen. Ing.-Arch. 12, 109–123 (1941).

The author considers the motion of an inextensible cord. The cord consists of two parallel straight portions joined by a semicircular portion, so that the whole cord forms a letter U. The two straight portions are, however, not necessarily equal in length. The free ends are acted upon by forces in the direction of the straight portions of the cord. The cord is idealized to the point where the area of the cross section is zero and the radius of the circular portion is zero, so the two straight portions lie in the same straight line. The system then has two degrees of freedom, and the two equations of motion are readily deduced from Lagrange's equations of motion. These two equations are then solved in several special cases in which either the forces acting on the free ends or the displacements of the free ends are specified. *G. E. Hay* (Ann Arbor, Mich.).

de Sz. Nagy, Béla. Vibrations d'une corde non homogène. Bull. Soc. Math. France 75, 193–208 (1947).

This treatment of the vibrations of a nonhomogeneous string utilizes the spectral theory of completely continuous self-adjoint operators in Hilbert space. Standard results on the nature of the normal vibrations and the expansion problem for general vibrations are extended to the general case of an increasing mass function. The final motivation for the abstract operator technique, however, is the application of the perturbation theory of self-adjoint operators developed by Rellich and extended by the author [Comment. Math. Helv. 19, 347–366 (1947); these Rev. 8, 589] to obtain explicit estimates for the rapidity of convergence of the (Rayleigh) perturbation series corresponding to a small change in mass function. The case of a homogeneous string with a point load is worked out in detail.

W. F. Eberlein (Madison, Wis.).

Facciotti, Guido. Deduzione cinematica diretta del teorema di Coriolis. Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. (3) 10(79), 198–200 (1946).

Castoldi, Luigi. Sopra una definizione di "spostamento virtuale" di un sistema dinamico valida per vincoli di natura qualunque. Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. (3) 10(79), 71–88 (1946).

Hydrodynamics, Aerodynamics

Nel'son-Skornyakov, F. B. Filtration through an embankment and a dam with a core or a screen (impervious foundation). Doklady Akad. Nauk SSSR (N.S.) 60, 1137–1140 (1948). (Russian)

This paper deals with the problem of filtration of water through a dam with an impervious core of triangular cross section of variable shape. The velocity potential and the flux are derived. From these the lines of flow can be constructed and the loss of water through the dam computed. A comparison is made between the results obtained from the exact formulae with those from an empirical hydraulic formula. *A. W. Boldyreff* (Albuquerque, N. M.).

Traupel, W. On the potential theory of blade cascades. Sulzer Tech. Rev. 1948, no. 2, 12–30 (1948).

A theoretical contribution to the potential theory of the blade cascade has appeared [same Rev. 1945, no. 1, 25–42; these Rev. 7, 423]. The present article is a continuation which permits the conditions of flow to be plotted comprehensively for any inlet angle.

From the author's summary.

Fainzil'ber, A. M. On a problem of chemical dynamics. Doklady Akad. Nauk SSSR (N.S.) 62, 457–460 (1948). (Russian)

The viscosity of an incompressible colloidal solution is taken to depend in a specific manner on the concentration. By suitable changes of variable the boundary layer type flow of such a solution can be reduced to the conventional boundary layer flow problem associated with a fluid of constant viscosity. The pressure gradient is taken to vanish in the analysis. *G. F. Carrier* (Providence, R. I.).

Fainzil'ber, A. M. Solution of the equations of isothermal gas flows by quadratures. Doklady Akad. Nauk SSSR (N.S.) 62, 603–606 (1948). (Russian)

By assuming isothermal flow, a particular case of the boundary layer equations for a compressible fluid is solved by quadratures. The result is not interpreted in terms of a specific physical situation. *G. F. Carrier.*

Sears, W. R. A new treatment of the lifting-line wing theory, with applications to rigid and elastic wings. Quart. Appl. Math. 6, 239–255 (1948).

The lifting-line theory of wing aerodynamics is so formulated as to lead to a nonhomogeneous integral equation of the second kind with fixed finite limits: the kernel is symmetrical and of integrable square and depends on the planform but not on the distribution of incidence along the span. The author proposes the laborious approach of calculating the eigenvalues and eigenfunctions, since if these were known for a widely used planform all problems involving that planform, even those which involve the elasticity of the wing, could be solved fairly simply by calculating from known eigenfunction expansions. [The question whether such expansions will converge rapidly enough for practical use is not discussed.] For trapezoidal wings of taper ratios 1, 2, 3, 3.5 and 4 the first ten eigenvalues are given and also five Fourier coefficients of each of the first ten eigenfunctions. *M. J. Lighthill* (Manchester).

***Burgers, J. M. A mathematical model illustrating the theory of turbulence.** Advances in Applied Mechanics, edited by Richard von Mises and Theodore von Kármán, pp. 171–199. Academic Press, Inc., New York, N. Y., 1948. $6.80.

The author considers a mathematical model of a hydronamic system to illustrate the theory of turbulence. The paper is a survey and an extension of the work presented by him in a well-known series of earlier publications. Although the system adopted is very simple, it is capable of illustrating many of the main features of turbulent motion, such as the thin dissipation layers arising from the small kinematic viscosity, the transfer of energy through the spectrum to high frequencies, and the appearance of a practical limit to this spectrum (responsible for the finite value of the total dissipation). A process of energy balance very similar to that in actual turbulent motion is thereby represented. In particular, the recent results on the correlation function $R(r)$ $[1-R(r)\sim r^{2/3}]$ and on the spectrum $E(k)$ $[E(k)\sim k^{-5/3}]$ at large Reynolds number are also reproduced in the model system. The main part of the paper deals with a model system having one component of turbulent motion; an extended model having two components is also discussed. *C. C. Lin* (Cambridge, Mass.).

Pekeris, C. L. Stability of the laminar parabolic flow of a viscous fluid between parallel fixed walls. Physical Rev. (2) 74, 191–199 (1948).

The author studies the stability of the laminar parabolic flow of a viscous fluid between parallel fixed walls. Asymptotic expressions are derived for the characteristic values C of two-dimensional disturbances of the parabolic flow in the limit of small wave numbers α of the disturbance and large values of αR, where R denotes the Reynolds number of the main flow. For the first mode, the author obtains

$$(1)\quad C_1=1.617e^{i\pi/8}(\alpha R)^{-\frac{1}{2}}+\alpha^2[.395-.883e^{i\pi/8}(\alpha R)^{-\frac{1}{2}}+(i/\alpha R)]+O(\alpha^4).$$

From this asymptotic formula, he infers the stability of the first mode and criticizes the work of Meksyn and of the reviewer, both of which indicate instability of the flow at sufficiently high Reynolds numbers. [Reviewer's remark. The second term in formula (1) is at least of the same order as the first term in the region where the reviewer predicts instability, i.e., in the region where $\alpha^3=O(\alpha R)^{-\frac{1}{2}}$ or larger. By examining the way the terms of various orders in (1) depend on αR, it might be conjectured that the next term would be of the order of $(\alpha R)^{\frac{1}{2}}$ and therefore not negligible in the region of instability. It would be interesting to have a very careful numerical check of the validity of (1) (with two terms only) with increasing αR. The above discussion would suggest its failure long before the instable region is reached.] Other modes are investigated, and the conclusion of stability is reached, provided formulae of the type (1) are used (i.e., characteristic values calculated only up to terms in α^2). *C. C. Lin* (Cambridge, Mass.).

Goldstein, S. On laminar boundary-layer flow near a position of separation. Quart. J. Mech. Appl. Math. 1, 43–69 (1948).

The author considers the singularity of the solution of the boundary layer equations at a position of separation. In particular, the case of linearly decreasing pressure gradient is considered. It is concluded that whenever this type of singularity occurs, the boundary-layer equations cease to be valid at and near separation on the upstream side, and also downstream of separation. The formulae found may help computers of laminar boundary layers to have an end point at which to aim. Mathematically, the work suggests that singularities of this nature may arise in the solution of nonlinear parabolic equations due to their nonlinearity. *C. C. Lin* (Cambridge, Mass.).

Howarth, L. Concerning the effect of compressibility on laminar boundary layers and their separation. Proc. Roy. Soc. London. Ser. A. 194, 16–42 (1948).

The author develops the theory of compressible flow in a laminar boundary layer for the case when the viscosity is assumed to be proportional to the absolute temperature and the Prandtl-number is unity. By a transformation of the ordinate normal to the layer and a modification of the stream function, it is shown that the equation of motion can be put into a simplified form very similar to the ordinary incompressible equation but modified by a multiplicative factor G in the pressure term. This factor is greater than 1 at the boundary and tends to 1 at the outside of the layer. Problems involving no pressure gradient can thus be solved by directly adopting the incompressible solution with suitable change of variables.

The momentum equation and the Kármán-Pohlhausen method are developed for the compressible boundary layer in this new system of variables. Methods of series expansion are also developed. The influence of pressure gradient is investigated by considering in detail flows with linearly increasing and decreasing mainstream velocity.

The general implications of the theory are discussed and qualitative conclusions are drawn when the mainstream velocity starts from a stagnation point, rises to a maximum and subsequently falls. It is concluded that for such a velocity distribution increasing compressibility will reduce the skin friction, increase the boundary layer thickness and cause earlier separation as compared with the incompressible flow with the same mainstream velocity distribution and the kinematic viscosity corresponding to conditions at the stagnation point. *C. C. Lin* (Cambridge, Mass.).

Howarth, L. The propagation of steady disturbances in a supersonic stream bounded on one side by a parallel subsonic stream. Proc. Cambridge Philos. Soc. 44, 380–390 (1948).

The author is interested in finding the way in which disturbances are propagated upstream in a boundary layer

when the mainstream is supersonic (e.g., interaction between a shock wave and a boundary layer) and other phenomena involving flows of both supersonic and subsonic speeds. The particular problem considered is a flow system consisting of a semi-infinite parallel supersonic stream (Mach number $M_1>1$) and an adjacent semi-infinite subsonic stream (Mach number $M_2<1$). General solutions are worked out; particular examples are also given. It is shown that the important parameter involved is

$$k=\gamma_2 M_2^2(M_1^2-1)/\{\gamma_1 M_1^2(1-M_2^2)^{\frac{1}{2}}\},$$

where γ_1 and γ_2 are the ratios of specific heats in the two streams. It is shown that a local initial disturbance in the supersonic stream generally produces a reflected wave extending to infinity both upstream and downstream of the initial disturbance. A (positive) pressure pulse as an incident wave produces pressure upstream and rarefaction downstream and the magnitude of the effects is calculated in a particular case. It is surmised that the effect of a shock wave on a boundary layer might be retarding both upstream and downstream of the shock wave. *C. C. Lin.*

Carrier, G. F. On the stability of the supersonic flows past a wedge. Quart. Appl. Math. 6, 367–378 (1949).

When a wedge of sufficiently small angle is placed in a uniform supersonic stream, an oblique shock starts at the vertex of the wedge. The solution is, however, not unique as there are generally two possible shocks for a given wedge. The shock with larger entropy increase is the "strong shock" and the flow after the shock is subsonic. The shock with smaller entropy increase is the "weak shock" and the flow after the shock is generally supersonic. The question the author tries to answer is the stability of these shocks, neglecting viscosity and heat conductivity, and assuming the wedge to be of infinite length. The problem is attacked by assuming small disturbances confined in the downstream region after the shock. The conditions at the shock are obtained by applying the Rankine-Hugoniot conditions to the motion of the shock caused by the disturbances. By separating the variables in a particular manner, the author shows that there is no characteristic oscillation which is nonsingular at the vertex of the wedge for both the strong shock and the weak shock. He then concludes that both shocks must be stable. It seems to the reviewer that the failure to find the proper characteristic oscillation could be the result of the method used to find the solution. Therefore, before a more definite answer to this question, the conclusion of stability can only be accepted as tentative.

In one section of the paper, the author calculates the unrelated problem of the interaction of a plane acoustic wave and a shock. *H. S. Tsien* (Cambridge, Mass.).

Cassen, B., and Stanton, J. The decay of shock waves. J. Appl. Phys. 19, 803–807 (1948).

The authors consider a general fluid motion depending on time and a Lagrange coordinate z which may relate to a one-dimensional, cylindrical or spherical type of flow. On the basis of a general equation of state, the gradients (with respect to z) of pressure and particle velocity in the wake of a shock are derived. These gradients become functions of shock velocity alone when account is taken of the equation of state (and an assumption concerning the time dependence of entropy in the wake is made). Thus, the conditions under which a shock will build up or will decay (i.e., the shock velocity increase or decrease) can be expressed in terms of a critical pressure gradient just behind the shock. If the influx of free energy from the explosive source is sufficient to keep the pressure gradient steeper than the critical, build-up will occur; in the contrary case decay. Actual cases of build-up (or more accurately non-decay) have been observed, and the authors discuss these in connection with a derivation of the higher z-derivatives of pressure in the wake.

As a further application, a time-dependent solution for a low Mach number ($M>1$), constant entropy plane flow is given, and it is shown that this solution can be terminated by two shocks. These shocks move apart at a rate proportional to the square root of time with a predictable rate of decay. *D. P. Ling* (Murray Hill, N. J.).

Thomas, T. Y. On the stability and instability of shock waves. Proc. Nat. Acad. Sci. U. S. A. 34, 526–530 (1948).

The author considers a plane uniform supersonic flow impinging on a pointed obstacle whose boundary (symmetric with respect to the flow direction) is an analytic curve. The attached curved shock line is likewise assumed analytic. At the vertex V, the tangent to the obstacle makes an angle ω with the flow direction, and the shock tangent makes an angle α with the obstacle tangent. The curvatures of shock and obstacle at V are denoted by k and K, respectively, and successive derivatives of these quantities with respect to the appropriate arc length are denoted by subscripts.

The author then displays a set of consistency relations between K_n and k_n ($n=0, 1, 2, \cdots$), the nth relation involving a function $G_n(M, \alpha)$, where M is the Mach number of the incident flow. When for given M and α any one of the G_n vanish, the corresponding configuration cannot persist, i.e., is unstable, and α is said to be singular relative to M. For every $M>1$ there is an angle $\beta(M)$ with the following property: no singular value of α exists in the interval $\alpha_0(M)<\alpha<\beta(M)$ ($\alpha_0(M)$ representing the Mach angle), while in the interval $\beta(M)\leqq\alpha\leqq 90°$ the singular values of α are dense. Specifically, $\beta(M)$ represents the angle at which the local flow at V just behind the shock changes from supersonic to subsonic. From this it can be deduced that any shock configuration for which α lies in the upper interval is unstable, whereas if α lies in the lower interval it is at least locally stable and presumably stable in the large.

From all permissible values of ω (i.e., those for which an attached shock is possible) there are two possible values of α. For the most part one of these lies in the stable, the other in the unstable, region. For values of ω near the maximum, however, both values of α appear to represent unstable solutions, even though the Rankine-Hugoniot conditions are in neither case violated. *D. P. Ling.*

Green, George. Waves in deep water due to a concentrated surface pressure. Philos. Mag. (7) 39, 738–743 (1948).

The author applies a method described by Kelvin [Proc. Roy. Soc. Edinburgh. Sect. A. 26, 399–432 (1906), p. 412] to two problems of two-dimensional motion of water initially at rest under gravity: (1) the waves due to the application of a time periodic surface pressure, (2) the waves due to a moving concentrated pressure applied for a time interval t. The author states that his result in the case of (1) disagrees with that given by Lamb [Hydrodynamics, 3d ed., Cambridge University Press, 1905, pp. 375–377, but omitted in later editions]. This reference is not available to the reviewer, but the disagreement, here ascribed to Lamb's intro-

duction of Rayleigh's dissipative forces, could explain its omission from later editions. In the case of the second problem Lamb gives the solution for $t \to \infty$ and this agrees with that found by the author when $t \to \infty$.

L. M. Milne-Thomson (Greenwich).

Bilinski, Stanko. Contribution to the dynamics of the cumulonimbus cloud. Hrvatsko Prirodoslovno Društvo. Glasnik Mat.-Fiz. Astr. Ser. II. 3, 29–51 (1948). (Croatian. English summary)

An explanation is given of the barograph trace during the passage of a cumulonimbus, viz., the sudden increase and subsequent fall of the pressure. It is first shown that a descending air current can originate in the region of heaviest precipitation of the cumulonimbus because friction of the air on the raindrops can overbalance the buoyancy of the warm ascending current. Then a possible field of flow in a cumulonimbus is derived, and it is shown that this field of motion leads to a pressure distribution with a pressure rise during the passage of the cloud. *B. Haurwitz.*

Elasticity, Plasticity

Grioli, Giuseppe. Struttura della funzione di Airy nei sistemi molteplicemente connessi. Giorn. Mat. Battaglini (4) 77, 119–144 (1947).

In plane strain and generalized plane stress in the absence of body forces, the problem of equilibrium for a homogeneous isotropic elastic body involves the determination of a biharmonic Airy function $F(x, y)$. The paper deals with the case where the body is multiply connected, F and its first derivatives being then multiple valued. The case of a doubly connected body (one hole) is treated in detail. Taking the origin in the hole, the author shows that F must have the form $F=\phi-\theta(M+yX-xY)/2\pi$, where ϕ is biharmonic, single valued and with single valued first derivatives, $\tan\theta=y/x$, and X, Y, M represent the resultant force and couple across any circuit drawn in the body and enclosing the hole. The next step is to express ϕ in the manner of Poincaré in the form

$$\phi=\phi_0+\phi_1+(\alpha\rho^2+\beta x+\gamma y+\delta)\log\rho+a\cos 2\theta+b\sin 2\theta,$$

where ρ, θ are polar coordinates, the coefficients are constants, and ϕ_0, ϕ_1 are biharmonic, the former regular in the portion of the plane inside the outer boundary of the body, and the latter regular in the portion of the plane exterior to the hole and converging at infinity. The general problem is then analyzed in two parts: (a) when there are no stresses across the boundary and the displacement is multiple valued, (b) when there are stresses across the boundary and the displacement is single valued. The meaning of the constants occurring in F is examined. Series developments for ϕ_0 and ϕ_1 are suggested. The method is extended to cover multiple connectivity of any order. *J. L. Synge* (Dublin).

Stevenson, A. C. The Dirichlet problem for a ring space. Philos. Mag. (7) 39, 297–303 (1948).

The author considers the determination of a function $\phi+i\psi$ which is analytic in a ring space S and which is such that ψ has given values on the boundary C of S. The region S is mapped conformally onto the region between concentric circles, and the required function is then found by the use of Laurent series. As examples, there is deduced the solution to the torsion problem for a beam the cross section of which is the region bounded by (i) confocal ellipses, (ii) eccentric circles. *G. E. Hay* (Ann Arbor, Mich.).

Hruban, K. A semi-infinite solid subjected to the pressure of a rigid body. Acad. Tchèque Sci. Bull. Int. Cl. Sci. Math. Nat. 45 (1944), 63–75 (1945).

Two-dimensional problems involving a semi-infinite elastic solid are considered. There is introduced the known solution to the problem which arises when such a solid is acted upon by a general concentrated load acting at a point on its boundary [Love, A Treatise on the Mathematical Theory of Elasticity, 4th ed., Cambridge University Press, 1934, chap. 9]. By means of the superposition principle the solution is then computed for the case when a very general complex load acts on the plane boundary. The distribution of load corresponding to the case when a rigid body with a plane face exerts normal forces on the plane boundary is then deduced, apparently by means of a certain amount of guesswork. Other cases are considered similarly, among which are those when the rigid body exerts on the plane boundary (i) a couple, (ii) tangential forces. The case when the rigid body is a right circular cylinder is also considered. *G. E. Hay* (Ann Arbor, Mich.).

Hruban, K. Stress in an elastic half-space loaded by a rigid plate. Rozpravy II. Třídy České Akad. 54, no. 6, 36 pp. (1944). (Czech)

A summary is reviewed above.

Parkus, H. Der wandartige Träger auf drei Stützen. Österreich. Ing.-Arch. 2, 185–200 (1948).

A beam rests on three equally spaced supports. The cross section is a rectangle which is very narrow, the longer edges being vertical. The length of the beam is of the same order as the height of the beam. A uniform load acts on the lower face of the beam. The problem is considered as a problem in generalized plane stress, which means that it can be reduced to the determination of a stress function F which is biharmonic in a rectangle and satisfies certain boundary conditions. The function F is determined in a standard way as an infinite series involving trigonometric and hyperbolic functions, by the use of Fourier series. Some numerical examples are presented, and the results are compared with those obtained by use of the elementary theory of bending of beams. *G. E. Hay* (Ann Arbor, Mich.).

Swainger, K. H. Large displacements with small strains in loaded structures. J. Appl. Mech. 15, 45–48 (1948).

Bodies such as plates and rods can be subjected to large displacements without causing large strains. The author considers such displacements. The method employed consists in guessing the principal part of the displacement, then determining the correction to this guess so that the equations of equilibrium are satisfied. *G. E. Hay.*

Cabannes, Henri. Application du calcul symbolique à l'étude de la dynamique des fils. C. R. Acad. Sci. Paris 226, 1165–1168 (1948).

A general elastic wire acted upon by a general force field is investigated. By a consideration of the equilibrium of a finite portion of the wire, integral equations of equilibrium are obtained. In some special cases these equations are solved by the use of integral transforms. *G. E. Hay.*

Belluzzi, Odone. Lo studio delle strutture costituite da lastre curve. Boll. Un. Mat. Ital. (3) **3**, 97–105 (1948).

Sengupta, H. M. On the bending of an elliptic plate under certain distribution of load. III. Bull. Calcutta Math. Soc. **40**, 53–63 (1948).

[For part I cf. same vol., 17–35 (1948); these Rev. **10**, 172. No part II seems to have been published.] Using the Green's function derived in part I the author gives the expression for the deflection of a clamped elliptic elastic plate loaded along the major axis. Most of the paper is concerned with the convergence of the series obtained and the continuity of the functions so represented.

G. F. Carrier (Providence, R. I.).

Wang, Chi-Teh. Nonlinear large-deflection boundary-value problems of rectangular plates. Tech. Notes Nat. Adv. Comm. Aeronaut., no. **1425**, 113 pp. (1948).

The author considers the large deflection of a rectangular plate; that is, the deflection has the same order of magnitude as the thickness of the plate. It is well known that the equations governing this problem can be reduced to two nonlinear differential equations of the fourth order involving the deflection w and a stress function F. The equations are called the von Kármán equations. The author expresses these equations in difference form, and outlines an iteration method of solving them. Values of w at the lattice points are first guessed. The values of F at the lattice points are then deduced from one of the difference equations. This entails the solving of a number of linear equations. By use of the other difference equation, improved values of w are deduced. The whole process is then repeated until the desired accuracy is attained. This method is applied to a square plate under a uniform normal load, the edges being simply supported. The lattice points used number 36. The results obtained agree fairly well with those obtained by others. The author also considers the use of the relaxation process when the number of lattice points is very large.

G. E. Hay (Ann Arbor, Mich.).

Reissner, Eric. Finite deflections of sandwich plates. J. Aeronaut. Sci. **15**, 435–440 (1948).

The plates in question consist of two thin face layers of material having high elastic moduli with a thicker core layer of softer material between them. The two face layers are assumed to be stressed uniformly with respect to their thickness. The core layer is assumed to offer resistance to transverse shear and normal stresses only. In other words, none of the three layers individually has bending stiffness, but the combination of the three can resist bending. The author derives the theory for the bending of such composite plates under the action of transverse loads under the assumption that the transverse deflection, while small, is still not small enough that the strains in the middle surface can be neglected. The resulting theory is expressed in terms of two nonlinear differential equations, each of fourth order, which are the same as the von Kármán equations for uniform plates, except that there are additional terms in the bending equation which have their origin in the deformability of the core material. The author analyzes the range of the deflection-thickness ratio within which the nonlinear terms in the theory can be safely neglected; this range is found to decrease as the core is made softer relative to the faces. The author carries out the solution of the linear buckling problem for the uniformly stressed simply supported rectangular plate on the basis of his equations.

J. J. Stoker (New York, N. Y.).

Hemp, W. S. On a theory of sandwich construction. Coll. Aeronaut. Cranfield. Rep. no. **15**, i+10 pp. (1948).

The assumption is made that the core has transverse shear and direct stiffnesses only. A fourth order differential equation is obtained for symmetric buckling of the face plates (plate on elastic foundation) and an eighth order equation for buckling of the composite sandwich in an antisymmetric mode. The critical load for the latter case is the same as that obtained by E. Reissner [see the preceding review] with a fourth order equation developed by neglecting direct transverse deformability. Symmetrical buckling of a circular cylinder is analyzed with bending moment-curvature equations which include flexibility in shear and transverse compression. The latter is actually neglected in obtaining the critical load. *D. C. Drucker.*

MATHEMATICAL PHYSICS

Optics, Electromagnetic Theory

***Duffieux, P.-M. L'Intégrale de Fourier et ses Applications à l'Optique.** Chez l'auteur, Faculté des Sciences, Besançon. xiv+232 pp. 800 francs.

Though full of shortcomings which invite criticism, this book is not to be dismissed as valueless. It consists of nine chapters and an appendix, the last being written jointly with G. Lansraux. Chapter I sets out the elementary formalism of Fourier series and integrals, in so far as they are immediate consequences from the definitions. Chapter II discusses Fourier series and integrals in particular cases, some of which find application later. Chapter III gives the real formalism of Parseval's theorem (no attempt at a strict proof is made) and its connection with optical imaging, while chapter IV gives the elementary formalism of complex Fourier theory. Chapter V states the connection between Fourier transforms and diffraction at infinity. No attempt at a justification is made, and the reader is asked to accept the correspondence of Fraunhofer diffraction theory with that of Fourier transforms as "une erreur commode et traditionelle."

Of the remaining chapters two deserve special mention. Chapter VII bears the heading "Parseval's and Plancherel's theorems and coherent imaging" but its contents do not fully meet the expectations thus raised. Chapter IX, entitled "Transmission of frequencies and resolving power," contains interesting ideas, but does not develop them beyond their most obvious consequences. This criticism can indeed be made of the whole book, which is interesting and stimulating but somewhat lacking in sustained thought. The author, who is obviously an enthusiast for his subject, deserves praise for the way he has tried to bring Fourier theory into more intimate contact with experimental optics. Mathematicians will find many incorrect statements about the legitimacy of the operations used, and no ordinary student could safely use the book except in conjunction with a textbook to supplement its logical deficiencies. To a

critically minded reader, however, it could serve as a valuable stimulus to original investigation. *E. H. Linfoot.*

Sansone, Giovanni. Sulle superficie ugualmente illuminate da una sorgente luminosa. Pont. Acad. Sci. Acta 9, 127–134 (1945).

P. Paoli is said to have determined [Opuscula Analytica, Livorno, 1786, pp. 168–173] all plane curves Γ whose points are equally illuminated by a light source O in the same plane as the curve. In terms of suitable rectangular coordinates in the plane of Γ, with origin at O, the differential equation of a curve Γ of equation $y=y(x)$ is

$$|[\text{grad } (x^2+y^2)^{-\frac{1}{2}}]\cdot n| = \frac{|x\,dy-y\,dx|}{(x^2+y^2)^{\frac{1}{2}}(dx^2+dy^2)^{\frac{1}{2}}}=1,$$

where n is the unit normal to the curve. In polar coordinates (ρ, θ) with pole at O the equation becomes separable. Its solutions are the circle $\rho=1$ and James Bernoulli's lemniscate $\rho^2=\cos(2\theta+c)$, c a constant. The author states that V. Fossombroni [Saggio di Ricerche Sull'Intensità del Lume, Arezzo, 1781] considered this and other problems of the same nature, and derived, without attempting to solve it, the differential equation satisfied by a surface S in three-dimensional space whose points are equally illuminated by a light source O. In suitable rectangular coordinates with origin at O, the differential equation of the surface S of equation $z=z(x, y)$ is

$$(*)\quad |[\text{grad } (x^2+y^2+z^2)^{-\frac{1}{2}}]\cdot n| = \frac{|xz_x+yz_y-z|}{(x^2+y^2+z^2)^{\frac{1}{2}}(1+z_x^2+z_y^2)^{\frac{1}{2}}}=1,$$

where n is the unit normal to the surface.

The author integrates equation (*) by first introducing polar coordinates in space (ρ, θ, φ) with pole at O, obtaining the equation

$$(\rho^2)^2+\tfrac{1}{4}(\partial\rho^2/\partial\theta)^2+(4\sin^2\theta)^{-1}(\partial\rho^2/\partial\varphi)^2=1.$$

Its solutions are $\rho^2=1$, a sphere, and $\rho^2=\cos[2\tau(\theta, \varphi)]$, where

$$\tau(\theta, \varphi)=(\sin\theta)\int_{\varphi}^{\varphi}\cos\lambda(\theta, \varphi)d\varphi+\int_{\theta_0}^{\theta}\sin\lambda(\theta, \varphi_0)d\theta+c,$$

c is a constant, $\lambda(\theta, \varphi)$ satisfies

$$\varphi-\sin^{-1}\frac{\sin\lambda(\theta, \varphi)}{[1-\sin^2\theta\cos^2\lambda(\theta, \varphi)]^{\frac{1}{2}}}=F(\sin\theta\cos\lambda(\theta, \varphi)),$$

and F is an arbitrary function. It is shown how the function F may be determined when the surface S is required to pass through a curve $\rho=\rho_0(\varphi)$, $0\leqq\rho_0(\varphi)\leqq1$, on the (x, y)-plane $(\theta=\pi/2)$. Lastly, the two special cases in which

$$\text{(a)}\qquad \varphi-\sin^{-1}\frac{\sin\lambda(\theta, \varphi)}{[1-\sin^2\theta\cos^2\lambda(\theta, \varphi)]^{\frac{1}{2}}}=\text{constant},$$

(b) $\sin\theta\cos\lambda(\theta, \varphi)=$constant are examined. Case (a) leads to the surface obtained by revolving around the y-axis the Bernoulli lemniscate $(y^2+z^2)^2-(z^2-y^2)=0$, $x=0$, which lies in the (y, z)-plane. *J. B. Dias* (Providence, R. I.).

Toraldo di Francia, Giuliano. Sulle superficie illuminate uniformemente da un'onda cilindrica. Pont. Acad. Sci. Acta 10, 135–141 (1946).

G. Sansone [see the preceding review] has determined the surfaces in three dimensions which are equally illuminated by a point source of light, that is, by a spherical wave. The author deals with the analogous question for a cylindrical wave. In suitable rectangular coordinates (x, y, z) with origin O on the axis of the cylindrical wave, the differential equation of the surface S of equation $z=z(x, y)$ is

$$(x^2+y^2)^{\frac{1}{2}}|[\text{grad } (x^2+y^2)^{-\frac{1}{2}}]\cdot n| = \frac{|xz_x+yz_y|}{(x^2+y^2)(1+z_x^2+z_y^2)^{\frac{1}{2}}}=1,$$

where n is the unit normal to the surface. Introducing cylindrical coordinates (ρ, φ, z), the nonlinear first order equation $(1-\rho^2)z_\rho^2-z_\varphi^2=\rho^2$ is obtained. Its solutions are $\rho=1$, a circular cylinder with the same axis as the cylindrical wave; $\rho=\sin(\varphi+c)$, c a constant, a circular cylinder one of whose generators coincides with the axis of the cylindrical wave; and

$$z(\rho, \varphi)=\rho\int_{\varphi_0}^{\varphi}\sinh\lambda(\rho, \varphi)d\varphi+\int_{\rho_0}^{\rho}\rho(1-\rho^2)^{-\frac{1}{2}}\cosh\lambda(\rho, \varphi_0)d\rho,$$

where $\lambda(\rho, \varphi)$ satisfies

$$f\Big(\rho\sinh\lambda(\rho, \varphi),\ \varphi-\frac{\rho\sinh\lambda(\rho, \varphi)}{[1+\rho^2\sinh^2\lambda(\rho, \varphi)]^{\frac{1}{2}}}\times F(\cos^{-1}\rho, [1+\rho^2\sinh^2\lambda(\rho, \varphi)]^{-\frac{1}{2}})\Big)=0,$$

f being an arbitrary function and

$$F(\theta, k)=\int_0^{\theta}(1-k^2\sin^2\theta)^{-\frac{1}{2}}d\theta,\qquad k^2\leqq1,$$

the elliptic integral of the first kind. The particular case in which $\rho\sinh\lambda(\rho, \varphi)=$constant$\neq0$ is examined, and leads to helicoidal surfaces [L. Bianchi, Lezioni di Geometria Differenziale, v. 1, Pisa, 1903, p. 235]. When the constant is zero, the sphere $z^2=1-\rho^2$, whose center is on the axis of the cylindrical wave, is obtained. *J. B. Dias.*

Toraldo di Francia, G. Sulla luce diffratta da innumerevoli aperture distribuite a caso su un diaframma opaco. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 4, 730–735 (1948).

The author gives a general discussion of the diffraction of light passing through a large number of holes in an opaque diaphragm. The (complex) amplitude G of the diffracted light is the sum of a large number of mutually independent random variables, in the author's formulation, and the central limit theorem is then applied to derive a Gaussian distribution for G. The physical significance of the parameters of this distribution is discussed. *J. L. Doob.*

Durand, Émile. Diffraction de la lumière par le demi-plan indéfini parfaitement noir. C. R. Acad. Sci. Paris 226, 1440–1442 (1948).

Fresnel's method, consisting in an application of Huygens's principle, gives in this case a correct intensity-value but an error $\pi/4$ of phase. The author shows how a solution of the cylindrical wave-equation $\psi_{xx}+\psi_{yy}-c^{-2}\psi_{tt}=0$ can be used to obtain correct predictions of both intensity and phase. *E. H. Linfoot* (Cambridge, England).

Durand, Émile. Diffraction de la lumière par une ouverture circulaire dans un écran noir. C. R. Acad. Sci. Paris 226, 1593–1595 (1948).

A solution of the Fraunhofer diffraction problem of plane waves meeting normally a circular hole in a plane opaque screen is constructed on the basis of the wave-equation $\psi_{xx}+\psi_{yy}+\psi_{zz}-c^{-2}\psi_{tt}=0$. A solution for the corresponding

problem where the waves meet a circular opaque screen is also obtained. *E. H. Linfoot* (Cambridge, England).

Mirimanov, R. G. The diffraction of a spherical electromagnetic wave by a circular disc. Doklady Akad. Nauk SSSR (N.S.) **61**, 617–620 (1948). (Russian)

Maggi's transformation of Kirchhoff's surface integral into a line integral extended all along the edge of the diffracting screen is applied to the diffraction of a spherical wave by a circular disc whose axis passes through the point source. For the boundary conditions the simplified assumption is made that the scalar potential and its normal derivative are zero on the dark side of the disc, while their values in the other parts of the disc plane are the same as in the absence of the disc. The classical result is thus found that in the geometrical shadow the axis of the disc is characterized by a remarkable illumination. An approximate formula is given for points near the axis.

G. Toraldo di Francia (Florence).

Brehovskih, L. M. The field of refracted electromagnetic waves in the problem of point radiation. Izvestiya Akad. Nauk SSSR. Ser. Fiz. **12**, 322–334 (1948). (Russian)

Let the point source be a vertical dipole, placed at the height z_0 with respect to a horizontal plane separating two media of wave numbers k_0 and k, respectively ($k/k_0=n$). The Hertz vector for this dipole may be written in the form:

$$(1)\quad \Pi_0=\frac{e^{ik_0R_0}}{R_0}=\frac{1}{2}\int_{-\infty}^{\infty}H_0^{(1)}(\xi r)e^{\pm(\xi^2-k_0^2)^{\frac{1}{2}}(z-z_0)}(\xi^2-k_0^2)^{-\frac{1}{2}}\xi d\xi,$$

the receiver being placed at the height z and at the distances R_0 from the dipole and r from its axis. By means of the integral representation of the Hankel function one sees that (1) is the superposition of plane waves having different angles of incidence on the limiting horizontal plane. Multiplying then each term in the integral by its corresponding Fresnel reflection or refraction coefficient one obtains the reflected and refracted fields, Π_1 and Π', respectively. It follows that $\Pi=\Pi_0+\Pi_1$ in the upper medium and $\Pi=\Pi'$ in the lower one. The resulting integrals may be evaluated in the case $k_0r\gg1$ by taking the asymptotic expansion of the Hankel function and then applying the method of steepest descent. A detailed discussion is needed in order to prevent the path of integration from passing through a pole of the integrand during its deformation. In every case, if E_r', E_z' indicate the components of the electric field at a point of the lower medium and E_r, E_z the components in the upper medium for the same r and $z=0$, one finds the following relations:

$$E_r'=E_re^{-ik_0z(n^2-\cos^2\chi)^{\frac{1}{2}}},\quad n^2E_z'=E_ze^{-ik_0z(n^2-\cos^2\chi)^{\frac{1}{2}}},$$

where $\tan\chi=z_0/r$. Two identical formulas may be proved for a horizontal dipole. The same relations are shown to remain valid also in the case $|n|\to\infty$, where a pole of the integrand approaches the stationary point of the path of steepest descent. *G. Toraldo di Francia* (Florence).

Hallén, Erik. On antenna impedances. Acta Polytech., no. 16=Trans. Roy. Inst. Tech. Stockholm **1947**, no. 13, 18 pp. (1947).

The author extends his original formula for the input impedance of a cylindrical antenna to give explicit expressions and tables of values for the coefficients involved in the second order approximations. These are expressed in terms of iterated sine and cosine integrals which have already been tabulated [E. Hallén, Trans. Roy. Inst. Tech. Stockholm **1947**, no. 12; these Rev. **10**, 151]. The values given here supersede a shorter and less accurate table published by C. J. Bouwkamp [Physica **9**, 609–631 (1942); these Rev. **5**, 163]. *M. C. Gray* (Murray Hill, N. J.).

Thermodynamics, Statistical Mechanics

Liénard, A. Quelques particularités des déplacements de l'équilibre et stabilité. Ann. Fac. Sci. Univ. Toulouse (4) **8** (1944), 1–58 (1947).

The object of this paper is to characterize the manner of variation of the extensive and intensive thermodynamic magnitudes during a displacement from equilibrium. The treatment is classical. The paper begins by developing a generalized form of the law of inertia for quadratic forms in which the variables are subject to linear homogeneous relations. In chapter I, conditions are established that a homogeneous state be stable. It is shown that the conditions that equilibrium be realized adiabatically or monothermally are the same, contrary to the situation regarding stability under the influence of external forces. Systems having several phases are considered. Because of the "viscosity" of chemical reactions, it is necessary to consider, following a displacement from equilibrium, a very short interval of time during which only the physical magnitudes vary, and also a longer interval of time during which chemical actions occur with further change in the physical magnitudes. The author shows that, in a number of different situations, the changes in extensive magnitudes have the same sign during the first and second time intervals, while the changes in intensive magnitudes have opposite signs during the two time intervals, but with the second change being smaller than the first. Monothermal and adiabatic displacements are considered. *C. C. Torrance* (Annapolis, Md.).

Dutta, M. On a treatment of imperfect gas after Fermi's model. Proc. Nat. Inst. Sci. India **13**, 247–252 (1947).

Suppose each molecule of an imperfect gas has a "rigid volume b of exclusion." The author develops an expression for the entropy of a body of this gas by dividing the volume V into space cells of volume b, and assuming that each cell is either vacant or occupied by one molecule. Cohesive forces are allowed for by an additive correction a to V. The results approximate the van der Waals theory.

C. C. Torrance (Annapolis, Md.).

Dutta, M. On a treatment of imperfect gas after Fermi's model. II. Proc. Nat. Inst. Sci. India **14**, 163–168 (1948).

This is an alternative treatment of the paper reviewed above. *C. C. Torrance* (Annapolis, Md.).

Einbinder, Harvey. Quantum statistics and the $\aleph$ theorem. Physical Rev. (2) **74**, 805–808 (1948).

The aleph theorem states that at absolute zero an ideal quantum gas either condenses or has a zero point energy.

L. Tisza (Cambridge, Mass.).

Valurib, E. A., and Spivak, G. V. On finding the distribution function for non-equilibrium systems in the presence of transfer phenomena. Akad. Nauk SSSR. Zhurnal Eksper. Teoret. Fiz. 18, 539–547 (1948). (Russian)

The kinetic equation for nonuniform gases is found by maximizing the entropy under the constraints that the number of particles, the energy, the momentum, the angular momentum and the currents of these quantities have prescribed values. The collision terms are obtained by assuming a single constant relaxation time. *L. Tisza.*

Rushbrooke, G. S., and Ursell, H. D. On one-dimensional regular assemblies. Proc. Cambridge Philos. Soc. 44, 263–271 (1948).

A study is made of one-dimensional linear assemblies, i.e., linear chains of systems (e.g., molecules), each having its own energy, as well as interaction energies with other systems of the chain. The latter interaction energies are linear, depend only on the natures of the pair of reacting systems, and their distance (measured in terms of the number of other systems separating them). The basis of the treatment is to find an explicit form of the grand partition function of the assembly (i.e., the Laplace generating function for the probabilities of its states). The method is to show that the latter function results from the solution of a system of linear homogeneous difference equations of a mathematically simple type. Various results of earlier authors appear as special cases. It is shown that the type of system considered cannot exhibit critical conditions (in the thermodynamic sense). The mathematical treatment is only suggested in outline, and essentially mathematical entities and proofs are given in physical rather than in mathematical terms. *B. O. Koopman* (New York, N. Y.).

BIBLIOGRAPHICAL NOTES

***Reissner Anniversary Volume. Contributions to Applied Mechanics. Edited by the Staff of the Department of Aeronautical Engineering and Applied Mechanics of the Polytechnic Institute of Brooklyn.** J. W. Edwards, Ann Arbor, Michigan, 1949. viii+493 pp. $6.50.

The volume contains 32 papers; those of mathematical interest will be reviewed separately in Mathematical Reviews.

***Library of the Tenth International Congress of Philosophy. Volume I. Proceedings of the Tenth International Congress of Philosophy (Amsterdam, August 11–18, 1948).** North-Holland Publishing Company, Amsterdam, 1949. ix+1259 pp.

Pages 698–799 are devoted to Symbolic Logic and Philosophy of Mathematics.

***Polska Akademia Umiejętności. Polish Academy of Sciences and Letters. Wykaz Prac z Działu Nauk Matematyczno-Przyrodniczych Wykonanych w Polsce w Okresie Okupacji Niemieckiej w Latach 1939–1945. List of the Works Achieved in the Field of Mathematics and Sciences in Poland During the German Occupation, 1939–1945.** Kraków, 1947. iii+289 pp.

Pages 219–237 contain abstracts of mathematical papers; the majority of the papers have subsequently been published.

Académie Serbe des Sciences. Publications de l'Institut Mathématique.

Volumes 1 and 2 appeared in 1947 and 1948, respectively. The contents are in English, French, German and Russian. The journal is published in Belgrade.

Canadian Journal of Mathematics. Journal Canadien de Mathématiques.

Volume 1, number 1 appeared in January, 1949. This is a quarterly, published for the Canadian Mathematical Congress by the University of Toronto Press.

Collectanea Mathematica.

Volume 1, number 1 is dated 1948. This is a semi-annual, published by the Seminario Matemático de Barcelona.

Comptes Rendus de l'Académie Bulgare des Sciences. Sciences Mathématiques et Naturelles.

Volume 1, number 1 is dated January–March, 1948. This is a quarterly, published in Sofia. It contains only papers in English, French or German.

Journal of the Mathematical Society of Japan.

Volume 1, number 1 is dated September, 1948. The Mathematical Society of Japan is a continuation of the mathematical part of the former Physico-Mathematical Society of Japan. The editorial office is at Tokyo University.

Journal of the Indian Society of Agricultural Statistics.

Volume 1, number 1 appeared in 1948; there are to be two numbers a year. The journal is published by the Society, at New Delhi.

Mathematische Nachrichten.

Volume 1, number 1 is dated May, 1948. This is planned as a monthly; it is published by Akademie-Verlag, Berlin, for the Forschungsinstitut für Mathematik der Deutschen Akademie der Wissenschaften zu Berlin and the Mathematisches Institut der Universität Berlin.

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